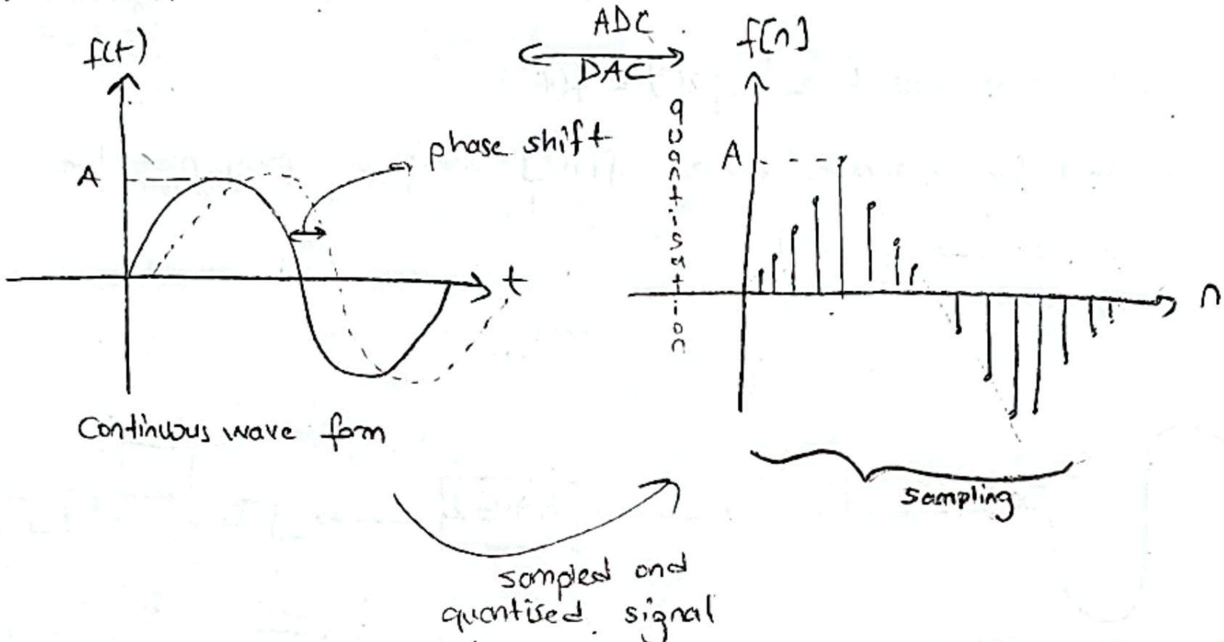
18/02/2026

EE 204

Signals and Systems

Defn: signal is a physical variation that carries information.



Signal is shaped by amplitude (A), phase shift and frequency $f = \frac{1}{T}$, where T is the period of the signal.

A → amplitude
↳ volume of the signal

ADC → Analog to Digital converter
DAC → Digital to Analog "

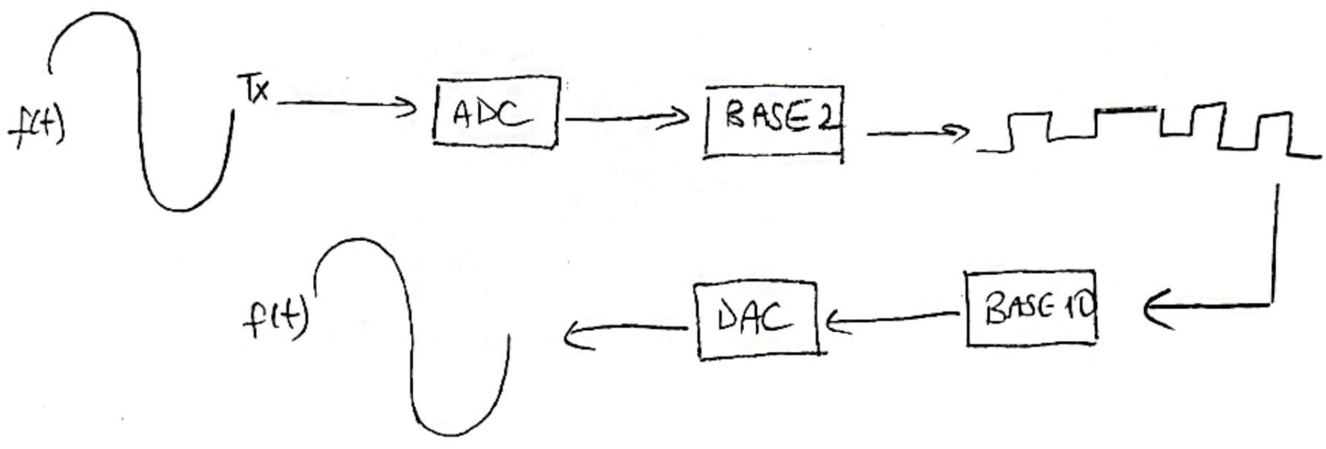
System: System is a HW and SW that takes the input and generates output using the signal.

$x(t)$ (INPUT) → $\boxed{\text{System}}$ → $y(t)$ (OUTPUT)

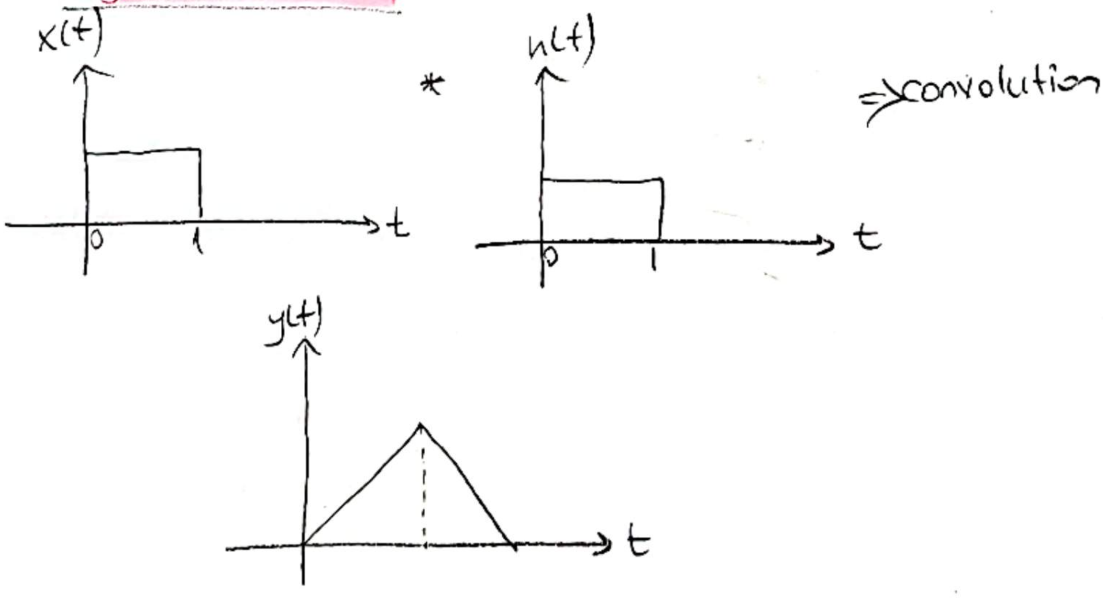
$y(t) = x(t) * h(t)$

↳ convolution generates output of a system. Using the input signal, in other words it gives the response of the system to the input signal.

* for continuous time $f(t^-) = f(t^+)$
* but for discrete time $f[n^+]$ may or may not be equal $\rightarrow f[n+1]$



Signal Transmission



Signal Energy and Power

Energy:

Sum of squares of abs value of signal between a given interval

$$E_c = \int_{t_1}^{t_2} |x(t)|^2 dt \Rightarrow P_c = \frac{E_c}{t_2 - t_1}$$

E_c: energy of continuous time signal

E_d = " " discrete time "

P: power

$$E_d = \sum_{n=-N}^{N} |x(n)|^2 \Rightarrow P_d = \frac{E_d}{2N+1}$$

Example: Evaluate the E_∞ and P_∞ of

a) $x_1(t) = \cos(t) \rightarrow$ cont. time

b) $x_2[n] = e^{j(\frac{\pi}{2n} + \frac{\pi}{8})} \rightarrow$ disc. time

Solu: a)

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \int_{-T}^{T} |\cos t|^2 dt = \lim_{T\to\infty} \int_{-T}^{T} \frac{1+\cos 2t}{2} dt$$

$$\lim_{T\to\infty} \frac{t}{2}\Big|_{-T}^{T} + \frac{\sin 2t}{4}\Big|_{-T}^{T} \overset{\lim_{T\to\infty}}{=} \underbrace{\frac{T}{2} + \frac{T}{2} + \frac{\sin 2T}{4} + \frac{\sin 2T}{4}}$$

$$\overset{\lim_{T\to\infty}}{=} T + \frac{\sin 2T}{2} = \infty$$

$\frac{\sin 2T}{2}$ → ignore because it is between -1 and 1.

$$P_\infty = \frac{E_\infty}{T-(-T)} = \frac{T + \frac{\sin 2T}{2}}{2T} = \frac{1}{2} + \frac{\sin 2T}{4T} = \frac{1}{2}$$

b) $$E_\infty = \sum_{n=-N}^{N} \sqrt{e^{j(\pi/2n + \pi/8)}}^{\,2}$$

Note: $|x(n)| = \sqrt{x(n) \cdot x^*(n)}$ ← conjugate

let $\pi/2n + \frac{\pi}{8} = \phi$ $\quad \sqrt{e^{j\phi} \cdot e^{-j\phi}} = \sqrt{e^0}$

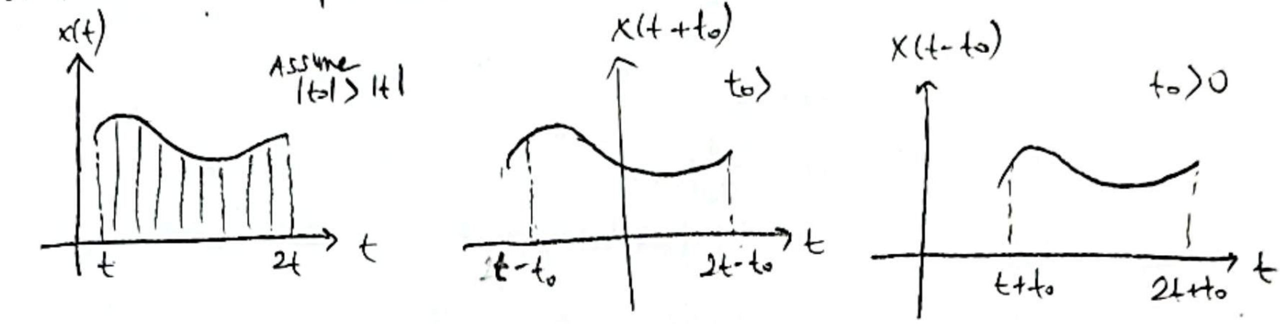
$$= \sum_{n=-N}^{N} \sqrt{e^0}^{\,2} = \sum_{n=-N}^{N} 1 = 2N+1$$

$$P_\infty = \frac{E_\infty}{2N+1} = \frac{2N+1}{2N+1} = 1$$

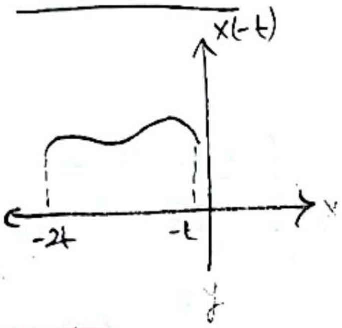
TRANSFORMATION OF INDEPENDENT VARIABLES

Time Shifting,

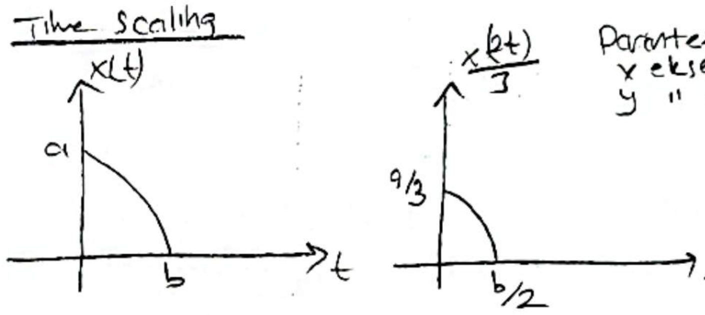
* First shifting must be done, if there also exists time scaling, or time reverse operations.



Time reversal



Time Scaling

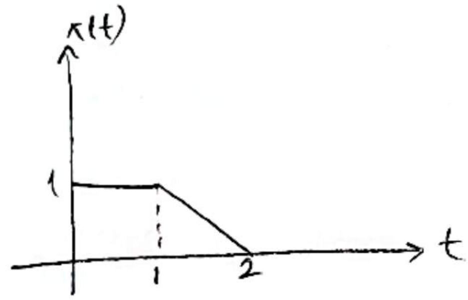


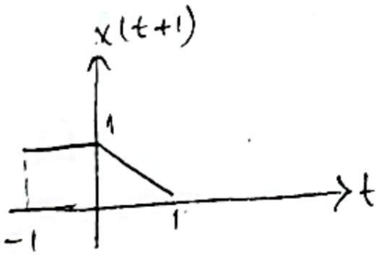
Parantezin içi x eksenini dışındaki y " etkiler

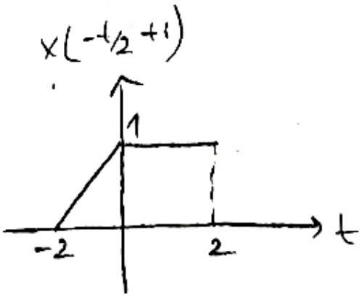
Example:

plot the graph of $x(-t/2+1)$ for the given $x(t)$ signal.

Önce shift yap daha sonra scale!
Ama unutup önce scale yaptıysak shift'i ters yönde yap!







Periodicity of the Signals

A signal is said to be periodic if and only if it has the same values for a specified period T. Note that we will only be able to use integer values for discrete time representation of the signals.

Example:

$x(t) = j \cdot e^{j10t}$ is the given signal periodic?

$\cancel{j}\, e^{j10t} \overset{?}{=} \cancel{j}\, e^{j10(t+T)}$

$\cancel{e^{j10t}} = e^{j10t} \cdot e^{j10T}$ $\boxed{e^{j2\pi m} = 1}$ $\longrightarrow$ $e^{j\alpha} = \cos\alpha + j\sin\alpha$

for $\alpha = 2\pi m$

$1 = e^{j10T} = e^{j2\pi m}$

$e^{j2\pi m} = \underbrace{\cos 2\pi m}_{1} + j\underbrace{\sin 2\pi m}_{0}$

$\boxed{e^{j2\pi m} = 1}$

$10T = 2\pi m$

$T = \frac{\pi m}{5}$ $m = 1, \dots,$ no fundamental period $\Rightarrow m = 1,\ T = \frac{\pi}{5}$

Example:

Check whether the given signal is periodic or aperiodic? (not periodic)

$x(t) = e^{(-1+j)t}$ $e^{j2\pi m} = 1$

$x(t+T) = e^{(-1+j)(t+T)}$

$\cancel{e^{(-1+j)t}} = e^{(-1+j)t} \cdot e^{(-1+j)T}$ $1 = e^{(-1+j)T} = e^{j2\pi m}$

$(-1+j)T = j2\pi m$

$T = \frac{j2\pi m}{(-1+j)}$

The signal is aperiodic because a Real T value (period) cannot be evaluated.

Example:

$x_2[n] = e^{j7\pi n}$

$x_2[n] = x_2[n+N]$

$e^{j7\pi n} = e^{j7\pi[n+N]}$

$\cancel{e^{j7\pi n}} = e^{j7\pi n} \cdot e^{j7\pi N}$ $1 = e^{j7\pi N} = e^{j2\pi m}$ $j2\pi m = j7\pi N$

$N = \frac{2m}{7}$

(it should be an integer) $m = 7 \Rightarrow N = 2$ (the fundamental period)

for an integer m

Example:

$$x_4[n] = 3.e^{j(3/5)(n+0,5)}$$

$$x_4[n] = x_4[n+N]$$

$$3.e^{j\frac{3}{5}\cdot[n+0,5]} = 3e^{j\cdot(3/5).[n+N+0,5]}$$

$$3.e^{j(3/5)[n+0,5]} = 3.e^{j(3/5)[n+0,5]} \cdot e^{j\frac{3}{5}N}$$

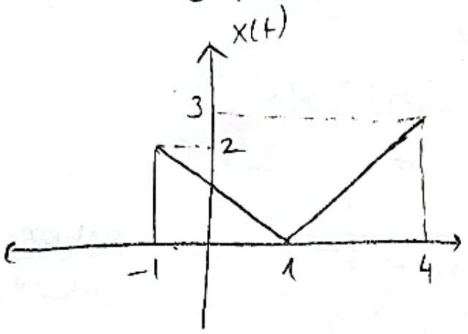
$$e^{j\frac{3}{5}N} = e^{j2\pi m} = 1$$

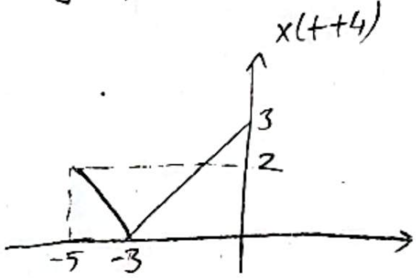
$$j\frac{3}{5}N = j2\pi m \qquad N = \frac{10\pi m}{3} \text{ (not periodic again)}$$

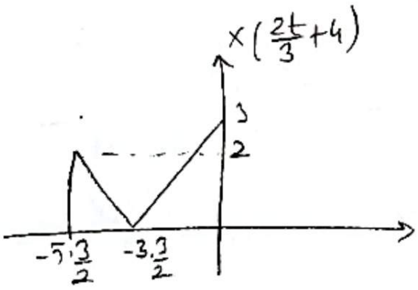
m = 1 ler için sayı olmuyor. π var

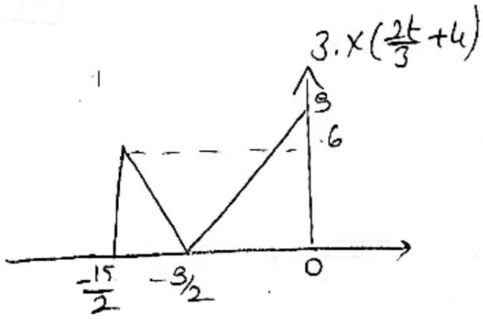
Example:

Plot the graph of $f(t) = 3.x\left(\frac{2t}{3}+4\right)$



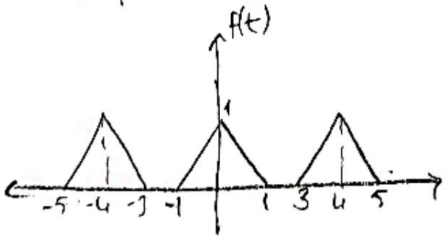


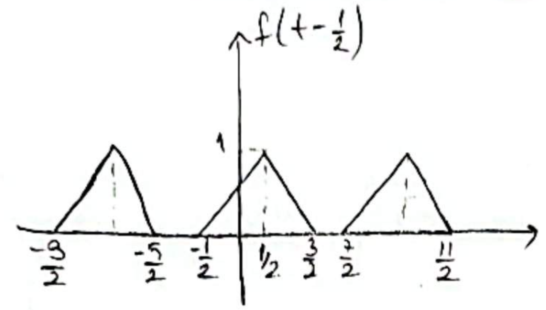


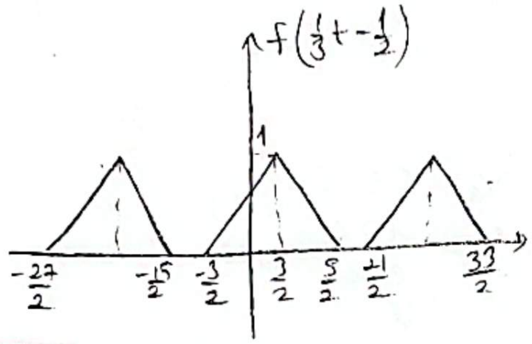


Example:

$g(t) = f\left(\frac{1}{3}t - \frac{1}{2}\right)$ is evaluated from the periodic signal $f(t)$ given, so plot the graph of $g(t)$

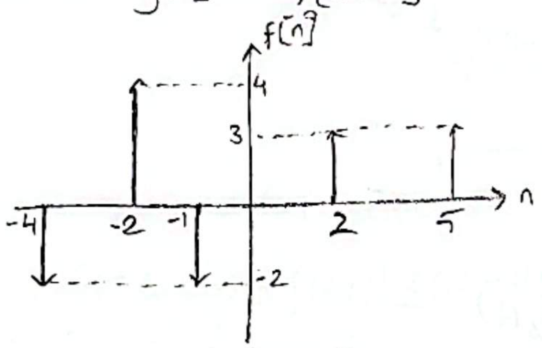


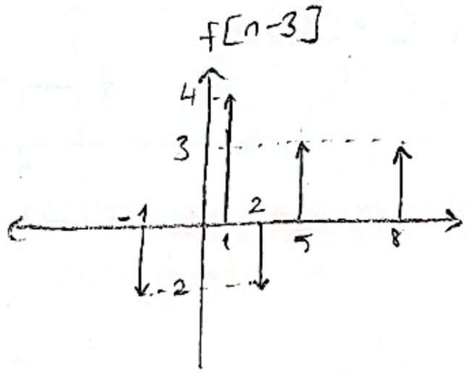


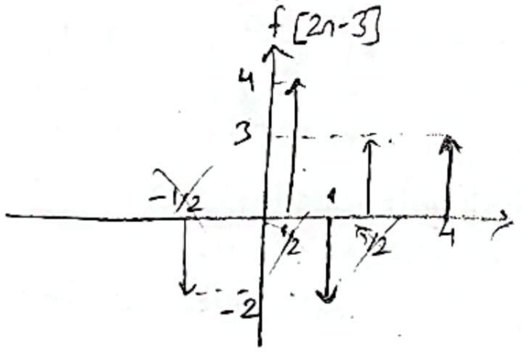


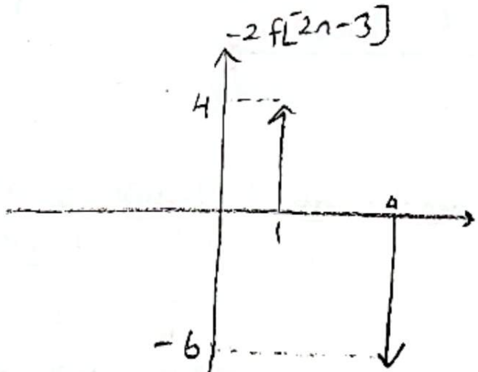
Example:

For the given $x[n]$ discrete time signal given below plot the graph for $g[n] = -2f[2n-3]$





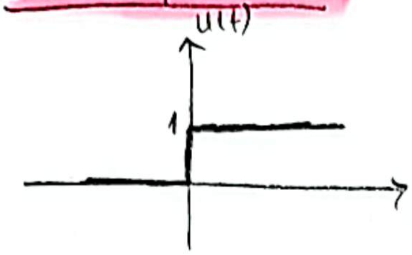




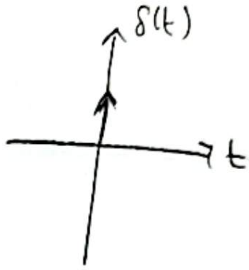
Signal Models in Telecommunications Area

All signals being used in telecommunications area can be represented in terms of some base signal models which are, unit step function, unit delta function, unit impulse function and delta impulse function.

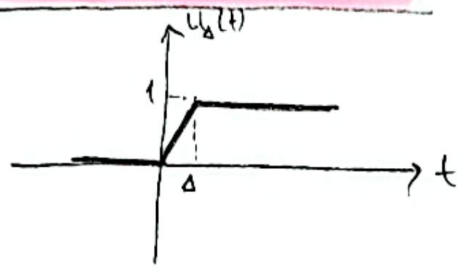
Unit Step Function



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \quad \xrightarrow{\text{türevi}} \quad \frac{d}{dt} u(t) = \delta(t)$$



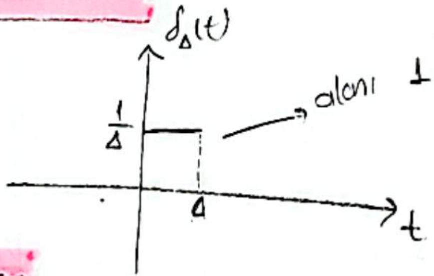
Unit Delta Step Function



$$u_\Delta(t) = \begin{cases} t/\Delta & 0 \leq t \leq \Delta \\ 1 & t > \Delta \\ 0 & \text{else} \end{cases}$$

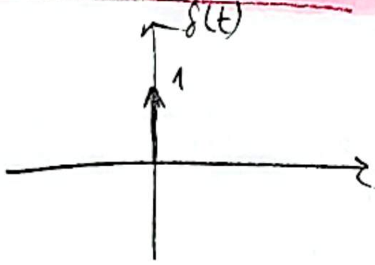
Unit Delta Impulse Function

$$\delta_\Delta(t) = \frac{d\, u_\Delta(t)}{dt}$$



→ alanı 1 olmalı.

Unit Impulse Function



$\delta(t) = \lim_{t \to \infty} \delta_\Delta(t)$ → ideal case $\Delta = 0$

$\frac{1}{\Delta} = \frac{1}{0} = \infty$, "ok ↑" sonsuza gider

ama işlemlerde 1 alınız. (çünkü alan hep 1'dir)

Unit impulse function has great sampling property to sample the continuous time signals and generate a discrete form.

(*) Bir sinyali impulse func. ile çarparsam o sinyalin sade "0" anındaki değerini alırım çünkü impulse func. da sadece 0 da değer var.

$$\sum_{n=0}^{\infty} x[n]\delta[t-n] \quad n=0 \text{ iken}$$

PROPERTIES OF IMPULSE FUNCTION

1) $\delta(t-t_0)f(t) = \delta(t-t_0)f(t_0) = f(t_0)$

2) $\delta(t-t_0)\cdot f(t-t_1) = \delta(t-t_0)\cdot f(t_0-t_1) = f(t_0-t_1)$

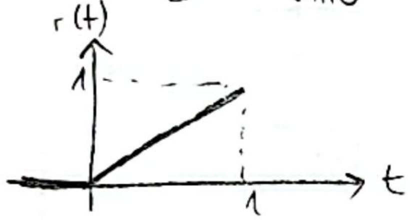
3) $\int_{-\infty}^{\infty} \delta(t)dt = 1$ OR $\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1$

4) $\int_{-\infty}^{\infty} \delta(t-t_0) f(t)dt = f(t_0)$, $\int_{-\infty}^{\infty} \delta(t-t_0)f(t-t_1) = f(t_0-t_1)$

RAMP FUNCTION

$r(t) = t \cdot u(t)$

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & \text{other} \end{cases}$$



$U(t) = \frac{dr(t)}{dt}$, $r(t) = \int_{-\infty}^{t} u(t)dt$

$\delta(t) = \frac{dU(t)}{dt}$, $u(t) = \int_{0}^{\infty} \delta(t)dt$

$\delta(t) = \frac{\partial^2 r(t)}{dt^2}$

$r(t) \xrightarrow{\text{derivative}} u(t) \rightarrow \delta(t)$, $\delta(t) \xrightarrow{\text{Integration}} u(t) \rightarrow r(t)$

Ex

If $f(t) = 2t^2+1$. Evaluate the results of the given operations

a) $f(t) \cdot \underbrace{\delta(t-1)}_{1} = \delta(t-1) \cdot f(1) = f(1) = 3$ Delta sadece içini 0 yapan yerde değer alır o da 1 dir.

b) $\int_{-\infty}^{\infty} f(t)\delta(t)dt = \int_{-\infty}^{\infty} f(t)dt = f(0) = 1$

c) $\int_{-\infty}^{\infty} f(t) \cdot \delta(t-2)dt = f(2) = 9$

Ex

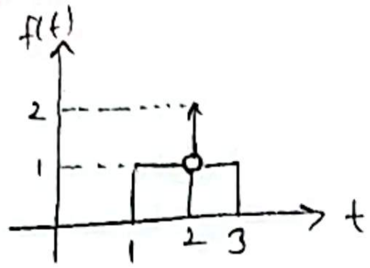
Evaluate the results of given expressions.

a) $\frac{du(t-2)}{dt} = \delta(t-2)$

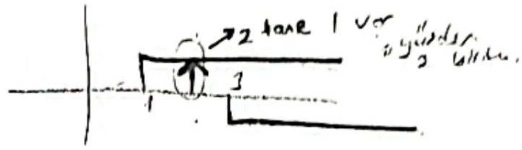
c) $\frac{dr(t^2+1)}{dt} = u(t^2+1)(2t)$

b) $\frac{du(t^2-1)}{dt} = \delta(t^2-1)2t$

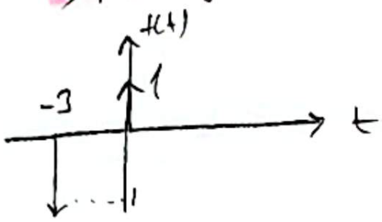
Ex: Draw the graph of given functions.



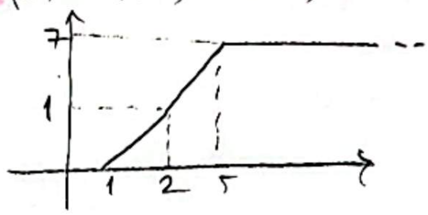
a) $f(t) = u(t-1) - u(t-3) + \delta(t-2)$



b) $f(t) = -\delta(t+3) + \delta(t)$



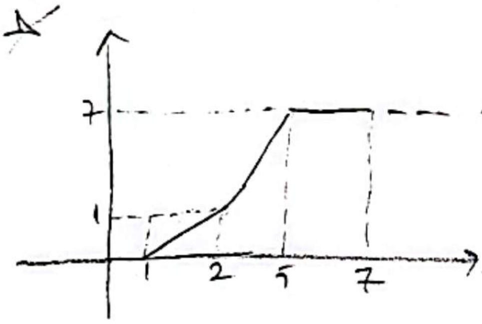
c) $f(t) = r(t-1) + r(t-2) - 2r(t-5)$



PROPERTY: $\delta(at) = \dfrac{1}{|a|}\delta(t)$ $\qquad \delta(a(t-t_0)) = \dfrac{1}{|a|}\delta(t-t_0)$

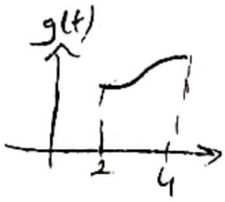
Ex: $\delta(2t) = \frac{1}{2}\delta(t)$ $\qquad$ **Ex:** $\delta(\underbrace{2t-1}_{2(t-1/2)}) = \frac{1}{2}\delta\left(t-\frac{1}{2}\right)$

Ex: $\delta(\underbrace{3t-3}_{3(t-1)}) = \frac{1}{3}\delta(t-1)$ $\qquad$ **Ex:** $\delta(-2t) = \frac{1}{2}\delta(t)$



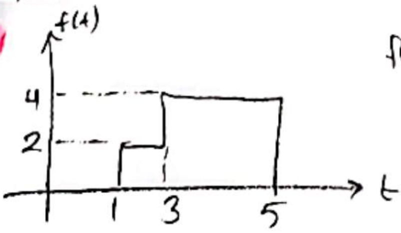
* 2 den öncesini yok etmek istersen $u(t-2)$ ile çarparım. 2 den öncesi 0 ile çarpılıp yok olur.

$g(t) = f(t)\cdot[u(t-2) - u(t-4)]$



Example: Write functions of the graph.

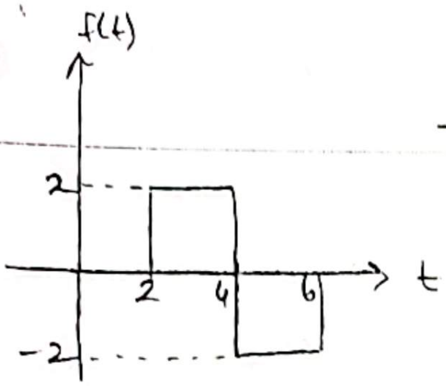
a)



$f(t) = 2[u(t-1) - u(t-5)] + 2[u(t-3) - u(t-5)]$

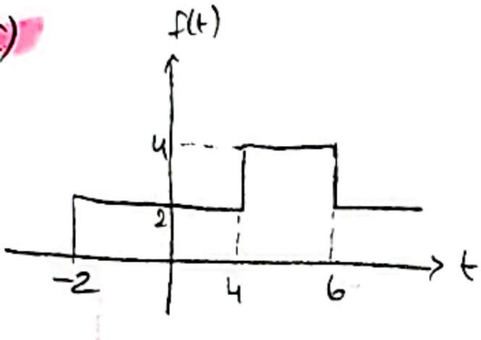
$= 2u(t-1) + 2u(t-3) - 4u(t-5)$

b)



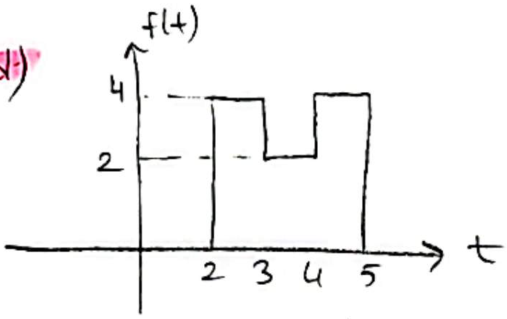
$$f(t) = 2[u(t-2) - u(t-4)] - 2[u(t-4) - u(t-6)]$$

c)



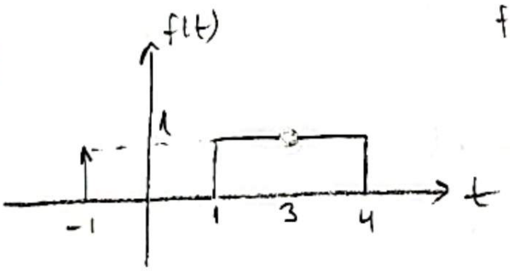
$$f(t) = 2u(t+2) + 2[u(t-4) - u(t-6)]$$

d)



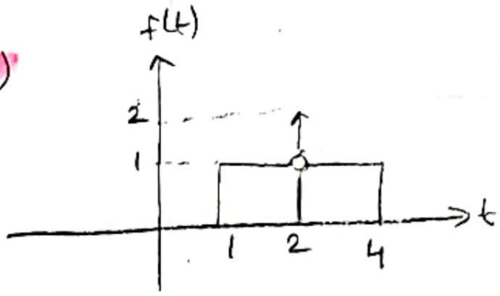
$$f(t) = 4[u(t-2) - u(t-5)] - 2[u(t-3) - u(t-4)]$$

e)



$$f(t) = u(t-1) - u(t-4) + \delta(t+1) - \delta(t-3)$$

f)

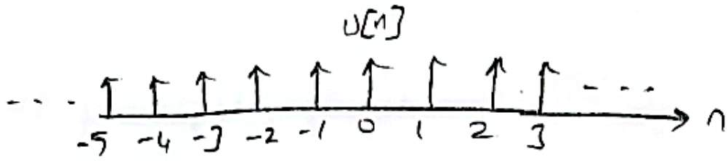


$$f(t) = u(t-1) - u(t-4) + \delta(t-2)$$

In discrete time signals;

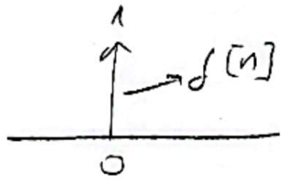
unit step function,

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{else} \end{cases}$$



Unit impulse function,

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{else} \end{cases}$$



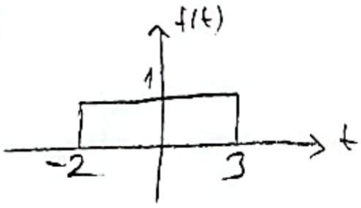
The relation between $u[n]$ and $\delta[n]$ is as follows

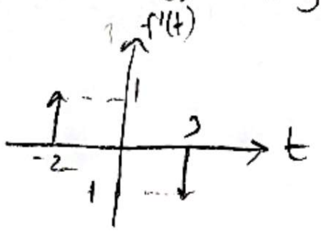
$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \qquad u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

same!

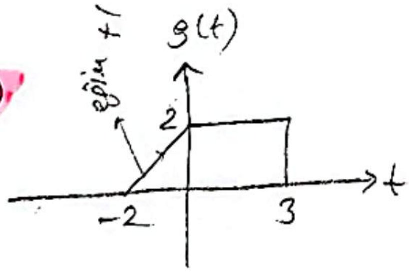
EX: Evaluate and plot the derivative of the given graphs.

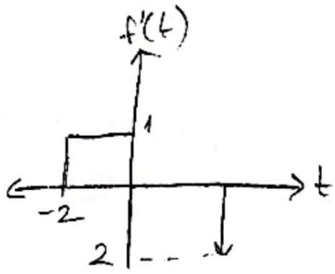
a)



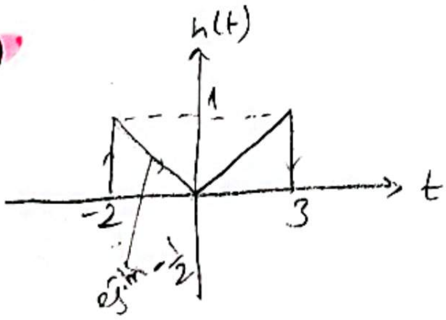


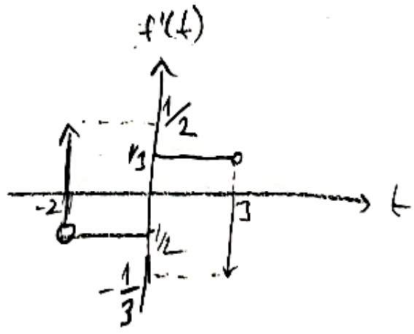
b)





c)





EVEN and ODD SIGNALS

Even

If $x(t) = x(-t)$ then the signal is said to be an even signal. The even signal is symmetrical to y-axis.

Odd

If $x(t) = -x(-t)$ then the signal is said to be odd signal. Odd signal symetrical to the origin point

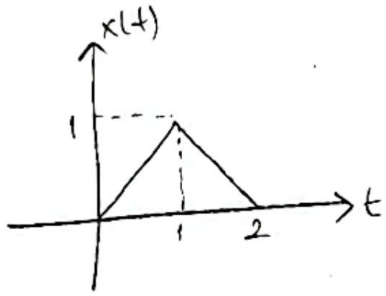
$$\text{Even}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)] \rightarrow \text{The even part}$$

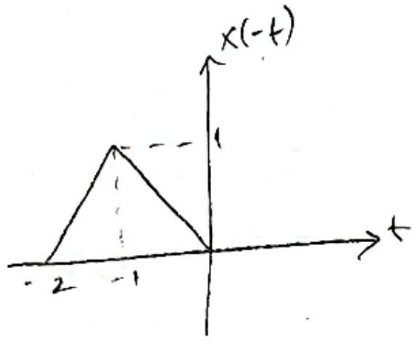
$$\text{Odd}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)] \rightarrow \text{The odd part}$$

Addition of the even part and the odd will give us the original signal itself.

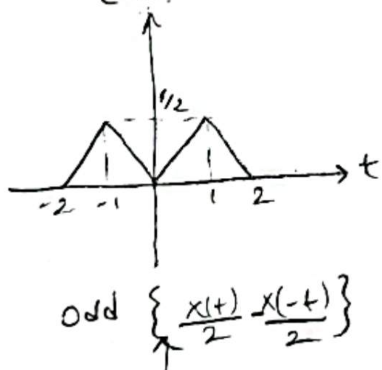
Example:

$$x(t) = \begin{cases} t, & 0 \le t \le 1 \\ -t+2, & 1 \le t \le 2 \end{cases}$$ find the even and odd parts of the signal.



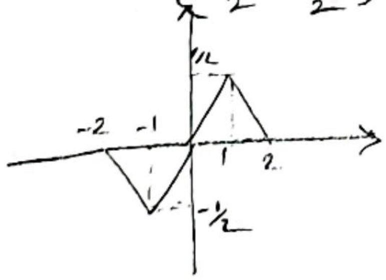


$$\text{Even}\left\{\frac{x(t)}{2} + \frac{x(-t)}{2}\right\}$$



⇒ This is the even part of the signal which is symetrical to the y axis.

$$\text{Odd}\left\{\frac{x(t)}{2} - \frac{x(-t)}{2}\right\}$$

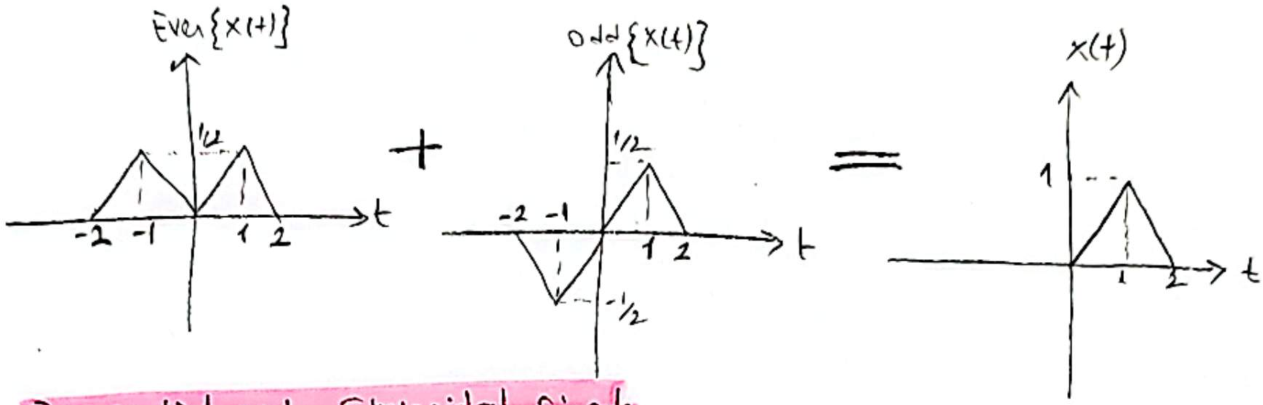


⇒ This is the odd part of the signal which is symetrical to the origin point.

$$x(t) = \text{Even}\{x(t)\} + \text{odd}\{x(t)\}$$

$$= \left[\frac{x(t)}{2} + \frac{x(-t)}{2}\right] + \left[\frac{x(t)}{2} - \frac{x(-t)}{2}\right]$$

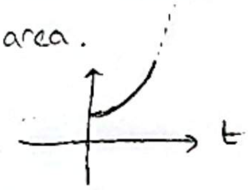
$$= \frac{x(t)}{2} + \frac{x(t)}{2} + \cancel{\frac{x(-t)}{2}} - \cancel{\frac{x(-t)}{2}} = x(t) \quad \text{So,}$$



Exponantial and Sinusoidal Signals

Exponantial signals are one of the most widely used signal types in telecommunication area.

$$f(t) = \begin{cases} c.e^{at} & t \geq 0 \\ 0 & \underbrace{\text{else}} \end{cases}$$



Because of this part $f(t)$ can also be written as;
$f(t) = c.e^{at}.u(t)$

On the other hand, sinusoidal function is defined as;

$$\boxed{f(t) = K.\cos\left(\frac{2\pi}{T}.t + \phi\right)}$$

$\frac{2\pi}{T} = W \rightarrow$ angular frequency

$T \rightarrow$ period of the signal

$\phi \rightarrow$ phase shift

$K \rightarrow$ amplitude

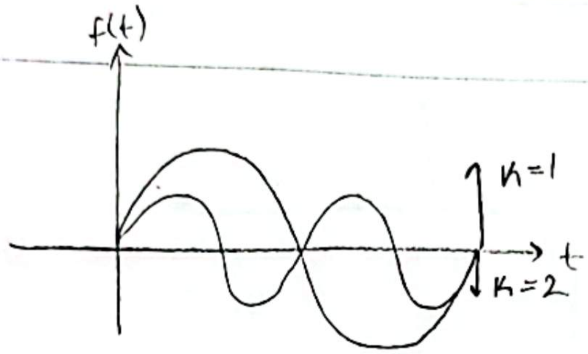
Example:

$f = \cos\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)$ find the period, angular frequency and the phase shift of the signal.

$W = \frac{\pi}{3} \quad \phi = \frac{\pi}{2} \quad w = 2\pi f \Rightarrow \frac{\pi}{3} = 2\pi f \quad f = \frac{1}{6} \quad T = 6$

* If there are more than one signals having common period. Then they are said to be harmonically related to each other

$\phi_k = e^{jk\omega t}$ where $k = 0 \pm 1 \pm 2 \pm 3 \pm \ldots$



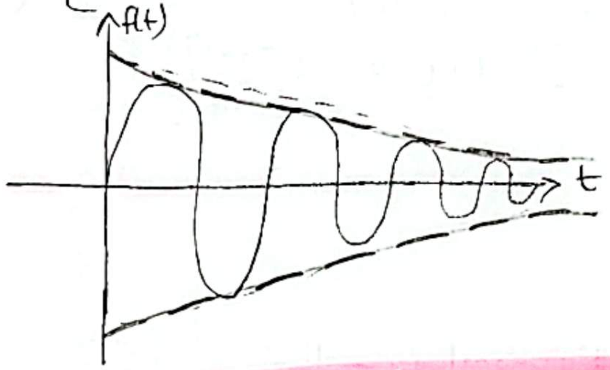
Sometimes we may also want to convert the sum of 2 exponential numbers to sum of multiplication of 2 simple complex number.

$$x(t) = e^{j2t} + e^{j3t}$$

$$= e^{j2,5t}\left(e^{j0,5t} + e^{-j0,5t}\right) = e^{j2,5t} \cdot 2\cos(0,5t)$$

* If a signal is multiplied by e^{-at} then the signal will be dumped.

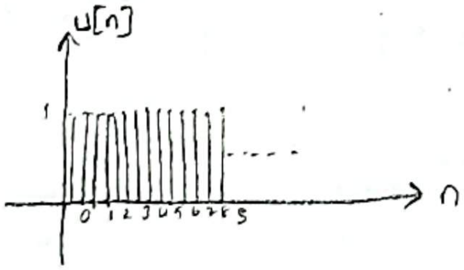
eg $f(t) = K \cdot e^{-at} \cos\left(\frac{2\pi t}{T} + \phi\right)$.



Discrete time Signal, Base functions:

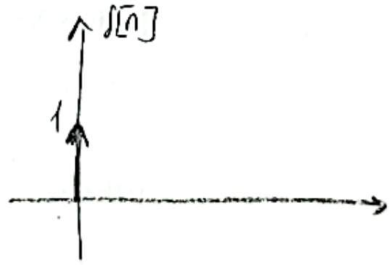
Unit Step function:

$$u[n] = \begin{cases} 1 & n \geq 0 \quad n \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$$

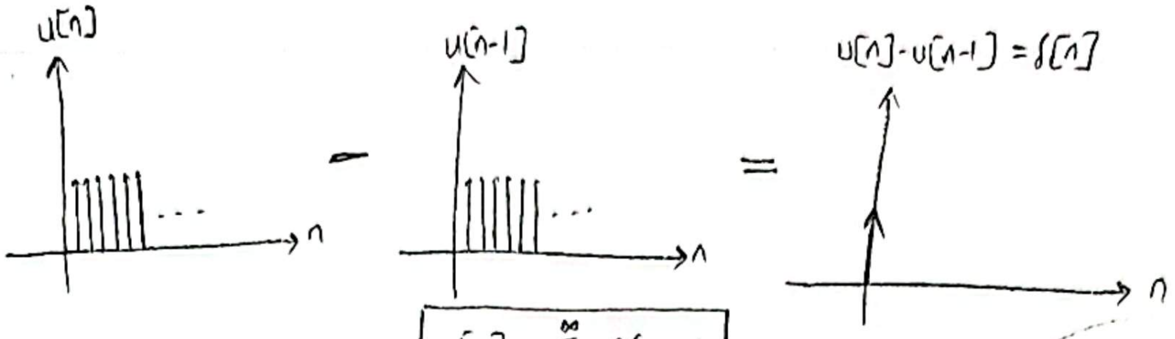


Unit Response Function:

$$\delta[n - n_0] = \begin{cases} 1 & n = n_0 \\ 0 & \text{else} \end{cases}$$

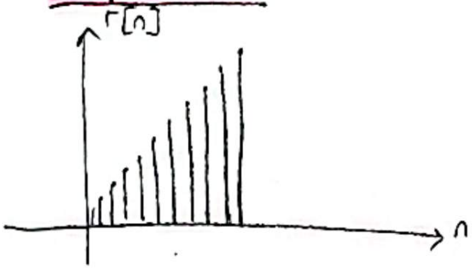


$$\delta[n] = u[n] - u[n-1]$$



$$\boxed{u[n] = \sum_{k=0}^{\infty} \delta(n-k)}$$

Ramp Function



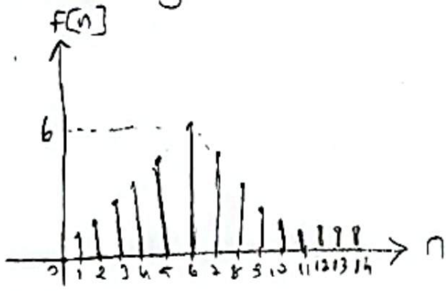
$$\boxed{r[n] = n . u[n]}$$

$$r[n] = \sum_{k=0}^{\infty} u[n-k] = \sum_{k=0}^{\infty} k . \delta[n-k]$$

Example:

Draw the graph of $f[n] = r[n] - 2r[n-6] + r[n-11]$

$$(r[n] - r[n-6]) - (r[n-6] - r[n-11])$$



We can also have the exponential signal in discrete form

$$f[n] = \begin{cases} k . e^{-n} & n \geq 0 \\ 0 & \text{else} \end{cases} \quad \Big\} \text{ same!}$$

$$f[n] = k . e^{-n} , u[n]$$

Comparison of discrete and continuous time signals for periodicity

Continuous Time

1) Takes different values for different w values.
2) It is periodic for any w_0 value.
3) The fundamental frequency is $\frac{2\pi}{|w_0|}$

Discrete Time

1) Takes the same values of w_0 values at multiples of 2π.
2) Periodic only for $w_0 = \frac{2\pi m}{N}$ ⇒ # of signals
3) $N = \frac{2\pi}{|w_0|} \cdot m$

Example

$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k]$ express this signal in terms of step function.

$k'+3$

$$\boxed{u[n] = \sum_{k=0}^{\infty} \delta[n-k]}$$ is known

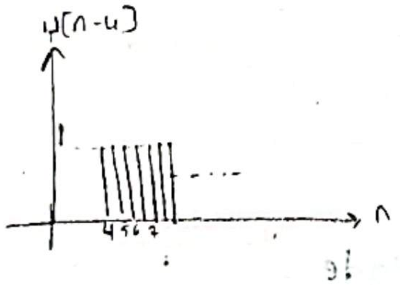
let's say $\underline{k' = k-3}$ then $x[n] = 1 - \sum_{k'=0}^{\infty} \delta[n-1-k'-3]$

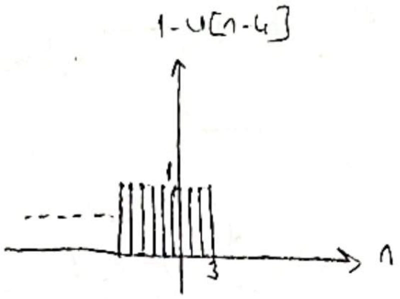
$x[n] = 1 - \sum_{k'=0}^{\infty} [n-4-k']$ let's say $\underline{n-4=n'}$

$x[n] = 1 - \sum_{k'=0}^{\infty} \delta[n'-k']$

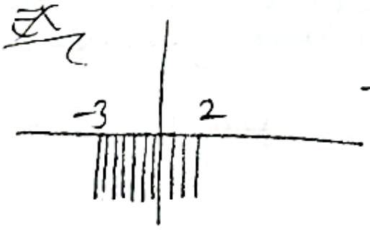
$x[n] = 1 - \sum_{k'=0}^{\infty} \delta[n'-k'] = 1 - u[n']$ by substituting the terms back

$x[n] = 1 - u[n-4] = u[-n+3]$





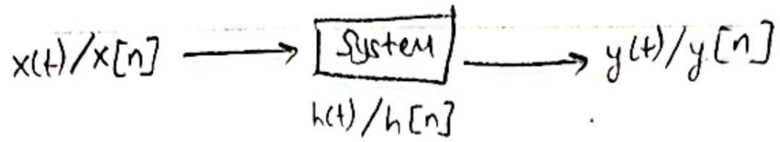
Ex

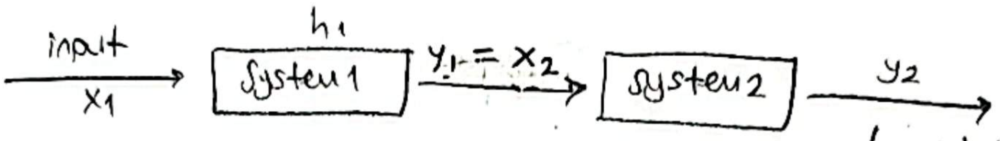


$-u[n+3] + u[n-3]$

Continious Time and Discrete Time Signals

In a system continious time or a discrete time signal can be taken as input signal, then the output is generated by the system using this input signal and the output is also generated in the same format.





(serial interconnection)

$x_1 * h_1 = y_1 = x_2$

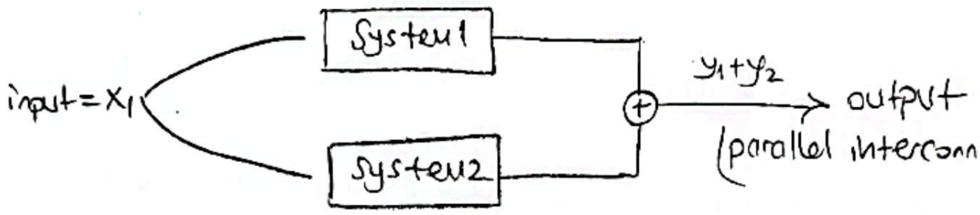
$x_2 * h_2 = y_2$

$y_2 = [\underbrace{(x_1 * h_1)}_{x_2} * h_2]$

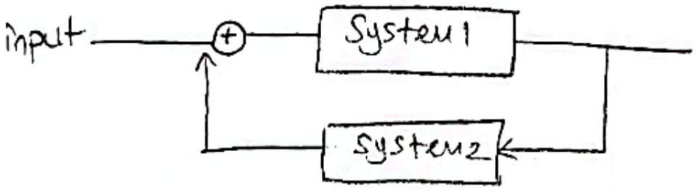
Ex

$S_1: \underbrace{y_1[n] = 2x_1[n] + 4x_1[n-1]}^{h_1[n]}$

$S_2: y_2[n] = \underbrace{x_2[n-1] + \frac{1}{2}x_2[n-3]}_{h_2[n]}$

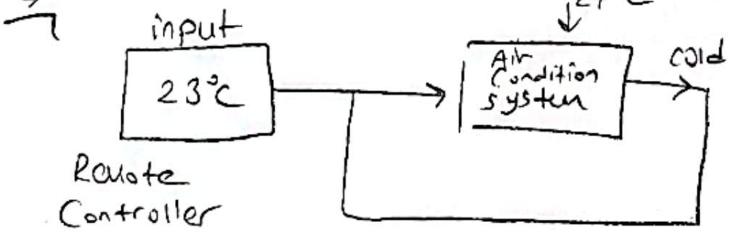


(parallel interconnection)



(feedback interconnection)

Ex



*Sistemden istenen 23°C sıcaklığı 27°C (havanın sıcaklığı ile karşılaştırıp gerekirse soğutuyor)

PROPERTIES OF THE SYSTEMS

Causality: If the output signal is related to current or past value of the input signal then the system is said to be causal

Ex: $y_1(t) = x(t-1)$ causal

$y_2(t) = x(t+1)$ not causal (geleceğe bağlı)

Memory: (To be memoryless or to have memory)

A system is said to be memoryless if and only if the output is related only the current time of the input signal.

$y(t) = x(t)$ memoryless

$y(t) = x(t+1)$ have memory

Invertability: The systems for which we can only determine distinct outputs for each input are called as "invertable systems"

$y(t) = 2.x(t)$
$x(t) = \frac{y(t)}{2}$ } invertable

input - output birebir olmalı!

$y(t) = x^2(t)$
$x(t) = \pm\sqrt{y(t)}$ } non-invertable

Example Is the given system casual?

$\boxed{y(t) = x(-t)}$ → non casual

→ $y(-3) = +3$ oldu ilerdeki bir değere ihtiyaç duyuyor

$y(t) = x(-t)$ X
$y(3) = x(-3)$ ~~casual~~ ✓
$y(-3) \neq x(3)$ X

$y(t) = x(t).\underbrace{\cos(t+3)}_{\text{input değil}}$ → $y(0) = x(0)\cos(3)$

→ input geleceğe bağlı değil bu yüzden casual.

Stability

A system is said to be stable if a bounded (finite) output value can be evaluated from an unbounded (infinite) input value.

$y(t) = t.x(t)$ not stable

$y(t) = e^{-x(t)}$ stable $x(t) > 0$

*input sonsuza giderken output constant olursa stable

Time Invariance

A system is said to be time invariant if the shift on an input signal also shifts the output signal with the same amount.

$x(t) \to$ input
$y(t) \to$ output

Example

$\boxed{y(t) = \sin[x(t)]} \rightarrow y(t-t_0) = \sin(x(t-t_0))$ T.I.

$y[n] = n\,x[n]$ not T.I.

$y(t) = x(2t)$ not T.I.

$y(t) = 2x(t)$ T.I.

Linearity

$a x_1(t) + b x_2(t) = a y_1(t) + b y_2(t)$

Ex for $y(t) = t \cdot x(t)$

$$\left.\begin{aligned} y_1(t) &= t \cdot x_1(t) \\ y_2(t) &= t \cdot x_2(t) \end{aligned}\right\} \quad y(t) = a y_1(t) + b y_2(t) = t\,[a x_1(t) + b x_2(t)] \quad \text{Linear} \checkmark$$

$t\,x_1(t) = y_1(t)$

$t\,x_2(t) = y_2(t)$

Ex

$y(t) = x(t) \sin t$

$$\left.\begin{aligned} y_1(t) &= x_1(t) \sin t \\ y_2(t) &= x_2(t) \sin t \end{aligned}\right\} \quad [a x_1(t) + b x_2(t)] \cdot \sin t = a y_1(t) + b y_2(t)$$

Lineer olması için karesi, küpü gibi değişkenler olmalı.

Linear Time Invarient System

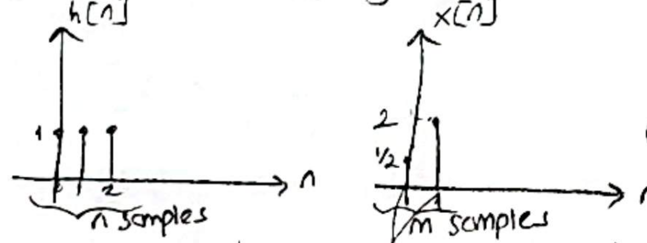
The convolution sum

Any discrete time signal can be represented in terms of shifted and weighted impulses.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] = \sum h[k] \cdot x[n-k]$$ is the representation of convolution sum.

Ex

Consider an LTI system with impulse response $h[n]$



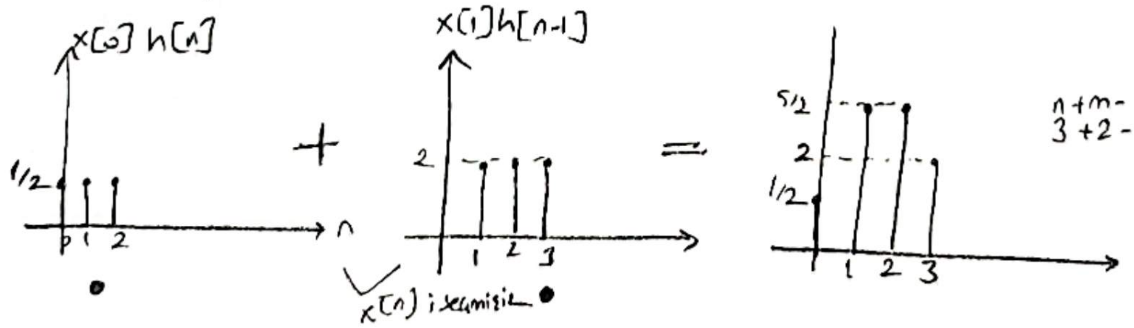
Convolution length will be $n+m-1$

$y[n] = x[n] * h[n]$

boyunu 1/2 ile çarp, kaydır.

boyunu 2 ile çarp

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[0] h[n] + x[1] h[n-1]$$



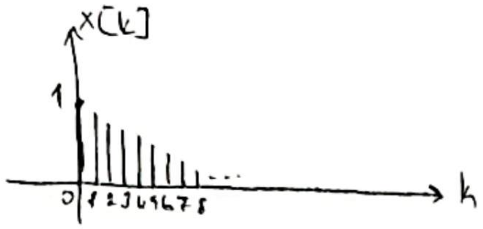
$n+m-1$

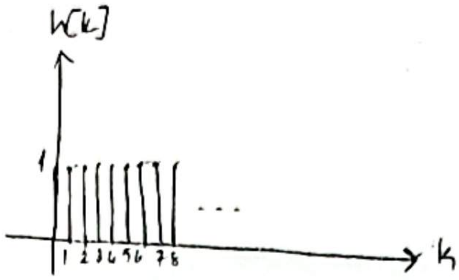
$3+2-1 = 4$

Ex

$x[n] = \alpha^n u[n] \quad 0<\alpha<1$

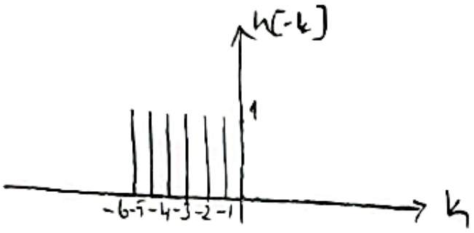
$h[n] = u[n]$

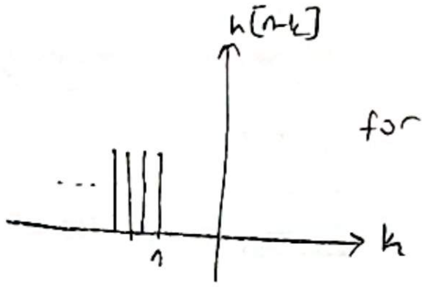




$y[n] = x[k] * h[k]$

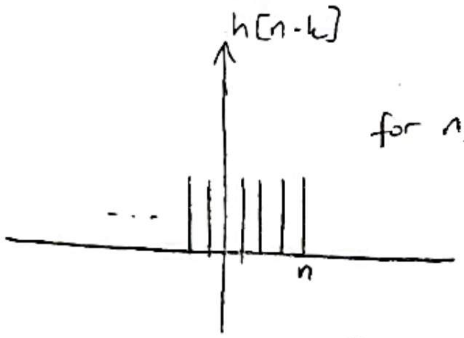
$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$



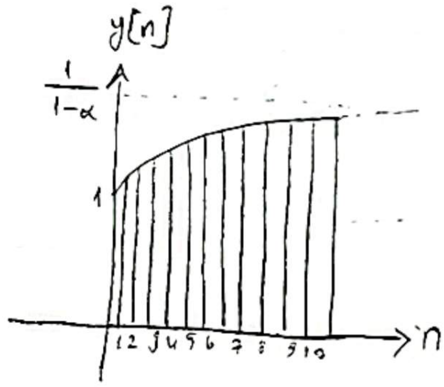


for $n<0$

$$\boxed{y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}}$$



for $n>0$



$0<\alpha<1$

Ex

$x[n] = [3 \; 5 \; 7 \; \underset{\Downarrow 0}{9} \; 2 \; -1 \; 6]$

$h[n] = [3 \; 5 \; 8 \; 10 \; \underset{\Uparrow 0}{3} \; 2 \; 1 \; 0 \; -2] \rightarrow$ 9 elemanlı (üsttekini 9 kere kaydırarak yaz)

$x[n] = [0 \; 3 \; 5 \; 7 \; \underset{\Uparrow}{9} \; 2 \; -1 \; 6] \rightarrow n$

$x[k]h[n-k]$

3x(0 0 3 9 5 15 7 21 9 27 2 -1 6)
5x(0 0 3 15 5 25 7 35 9 2 -1 6)
8x(0 0 3 24 5 40 7 9 2 -1 6)
10x(0 0 3 30 5 7 9 2 -1 6)
3x(0 3 5 7 9 2 -1 6)
2x(0 3 5 7 9 2 -1 6)
1x(0 3 5 7 9 2 -1 6)
0x(0 3 5 7 9 2 -1 6)
+ (-2)x(0 3 5 7 9 2 -1 6)

0 9 30 70 - - - sonuç 15 elemanlı $m+n-1 = 7+$

21

PROPERTIES OF CONVOLUTION

Commutative Property:

$$x[n] * h[n] = h[n] * x[n] = y[n]$$

Distributive Property:

$$x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$$

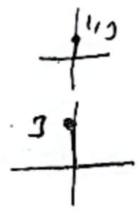
Associative Property:

$$x(t) * (h_1(t) * h_2(t)) = [x(t) * h_1[t]] * h_2(t)$$

LTI Systems with and without memory

Invertability of LTI systems

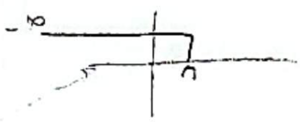
$$x(t) \longrightarrow \boxed{h(t)}_{1/3} \longrightarrow \boxed{h_1(t)}_{3} \longrightarrow y(t) = x(t)$$



$$x(t) \longrightarrow \overset{1}{\boxed{h(t) * h_1(t)}} \longrightarrow x(t)$$

$$h(t) * h_1(t) = \delta(t) \Rightarrow h(t) = 1/3, \quad h_1(t) = 3$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$



Example:

$h[n] = u[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,\overbrace{u[n-k]}^{1} \quad \text{for } n \geq k$$

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n-1} x[k] + x[n]$$

$$y[n] = y[n-1] + x[n] = \overbrace{y[n] - y[n-1]}^{\text{outputlardan}} = x[n]$$

outputlardan input bulabiliyorum sistem invertible

Since the $x[n]$ is evaluated back the system is invertible system.

Causality Property of LTI System:

To have an LTI system causal;

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{for } k>0, \quad h[n] \text{ must be } \underbrace{h[n]=0 \text{ for } n<0}_{\text{initial rest}}$$

initial rest
Yani 0 anına kadar input verilmemiş.

iki rest dedik 0 a eşitle y'leri bul

Because casual systems have to be related only on past or current time for instance $n=5$ and $k=7$ $y(5) = \underbrace{x(7)}_{\text{future}} h(5-7)$ $y[n] = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$ ↑ 7 ↑ 5

But if $h[n] = 0$ for $n<0$ this condition can be eleminated

Causal değil çünkü arabanın 5 anındaki hızı 7 anındaki hıza bağlı olmaz

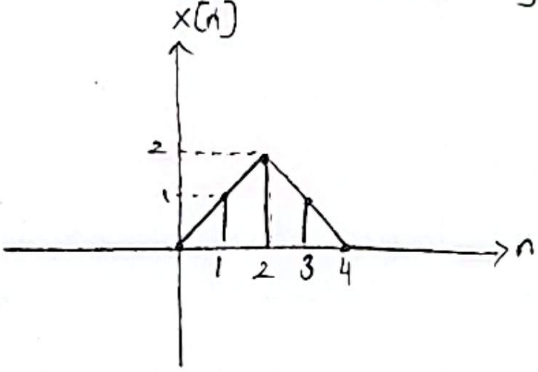
* A casual LTI system is equivalent to initial rest state. 5 anında sistem -2. değere bağlı

i.e. The output is also zero up to a certain point of time, if the input is zero up to that point.

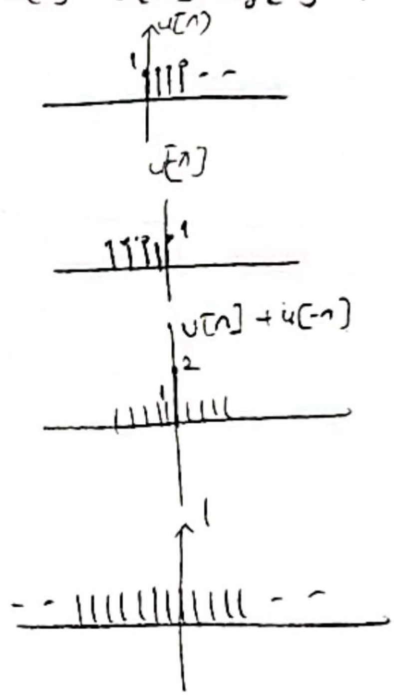
Bir sistem 0'dan önceki anlarda 0'a eşit.

EX

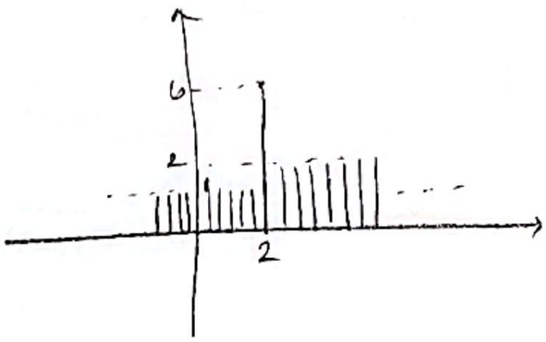
$x[n] = r[n] - 2r[n-2] + r[n-4]$



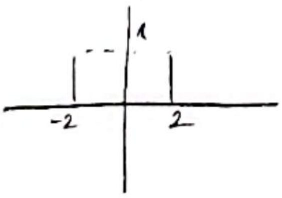
$u[n] + u[-n] - \delta[n] = 1$



$x[n] = \delta[n-2]n^2 + u[n-2] + 1$



$\delta(n^2-4)$



STABILITY FOR LTI SYSTEMS

$|x(n)| \leq B$ for all n

↳ Length of an input signal is bounded (absolute value)

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Since sum of absolute values of several numbers is always greater than or equal to sum of those numbers, we can write the expression,

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]| \quad \text{because } x[n] \text{ is bounded as } x[n] \leq B$$

LTI system is stable if and only if

$$\boxed{\int_{-\infty}^{\infty} |h[\tau]|\, d\tau < \infty}$$

Example

Consider a system that is a pure time shift in discrete time, is this system stable?

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n-n_0]$$

$$h[n] = \delta[n-n_0] \qquad k = n$$

$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ must be satisfied for the system to be stable.

$$\text{So, } \sum_{k=-\infty}^{\infty} |h[k]| = \underbrace{\sum_{k=-\infty}^{\infty} \delta[k-n_0]}_{\delta(n-n_0)} < \infty \qquad \text{Stable} ✓$$

Unit Step Response

Unit impulse can be evaluated by use of unit step function (response) which is denoted by $s[n]$ (or $s(t)$) if it is a continuous time signal). It is the output of the system when $x(t)$ or $x[n]$ is $u[t]$ or $u[n]$

for discrete time

$s[n] = u[n] * h[n]$

$$S[n] = \sum_{k=-\infty}^{\infty} h[k]\,u[n-k] \qquad n \geq k$$

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$\sum_{k=-\infty}^{n} h[k] = \underbrace{\sum_{k=-\infty}^{n-1} h[k]}_{s[n-1]} + h[n] \qquad s[n] = s[n-1] + h[n] \qquad h[n] = \underbrace{s[n] - s[n-1]}$$

türev demek bir önceki an ile bir sonraki an arasındaki fark.

for continuous time

$$h(t) = \frac{ds(t)}{dt} \qquad s(t) = \int_{-\infty}^{\infty} h(t)\,dt$$

→ cont. time da bir önceki ve sonraki anı alamıyorum çok yakın, bu yüzden fark değil türev alıyoruz.

Definition of Causal LTI Systems with Differential Equations and Difference Equations:

A continuous time signal can be expressed as linear constant coefficient differential equations.

$$\frac{dy(t)}{dt} + 2(y(t)) = x(t)$$

$$y(t) = \underset{\text{hom.}}{y_h(t)} + \underset{\text{particular}}{y_p(t)}$$

Homogenous Part

$$\frac{dy(t)}{dt} + 2y(t) = 0 \qquad y_h(t) = Ae^{st} \text{ is known}$$

Note that $y(t)$ is evaluated by summation of it's homogenous and particular parts. It will be called as the general solution.

$$sAe^{st} + 2Ae^{st} = 0 \implies Ae^{st}(s+2) = 0 \qquad s = -2$$

$$\boxed{y_h(t) = Ae^{-2t}}$$

Particular Part

If $x(t) = K.e^{3t}u(t)$

$Y_p(t) = Ye^{3t} \quad t \geq 0$

$3Ye^{3t} + 2Ye^{3t} = K.e^{3t}$

$5Y = K \quad Y = K/5$

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t} \qquad t \geq 0$$

For the causal system initial value is satisfied as zero for $t=0$ by using this property A can also be evaluated.

for causal systems output is zero for ~~$t<0$ if~~ $t=0$

$y(0)=0 \Rightarrow \quad A+\frac{K}{5}=0 \quad A=\frac{-K}{5}$

$y(t)=\left(-\frac{K}{5}e^{-2t}+\frac{K}{5}e^{3t}\right)u(t)$

$y(t)=-\frac{K}{5}e^{-2t}+\frac{x(t)}{5}$

Linear Constant Coefficient Differential Equns.

$y[n]-\frac{1}{2}y[n-1]=x[n]$

$y[n]=x[n]+\frac{1}{2}y[n-1]$

$x[n]=K\delta[n]$ (lets define)

and let the system be causal upto $n=-1$ -1 e kadar input yok.

$-\infty<n\leq-1 \quad x[n]=0 \quad y[-1]=0$ (initial rest)

causal

* initial rest olursa causal
Causal initial rest demek değil!

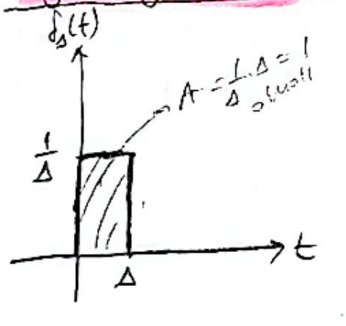
$y[0]=x[0]+\frac{1}{2}y[-1]=K+\frac{1}{2}\cdot 0=K$

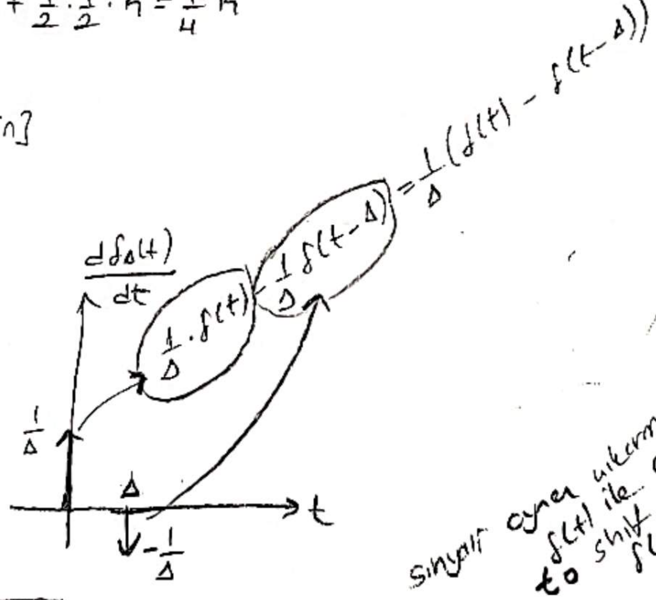
$y[1]=x[1]+\frac{1}{2}y[0]=0+\frac{1}{2}K$

$y[2]=x[2]+\frac{1}{2}y[1]=0+\frac{1}{2}\cdot\frac{1}{2}\cdot K=\frac{1}{4}K$

$y[n]=\left(\frac{1}{2}\right)^n K=\left(\frac{1}{2}\right)^n K\cdot u[n]$

Singularity Functions





sinyali aynen ulaşmak istiyorsak $\delta(t)$ ile conv. ederiz
t_0 shift etmek istersek $\delta(t-t_0)$ ile conv. ederiz.

$\frac{d\delta_\Delta(t)}{dt}=\boxed{\frac{1}{\Delta}[\delta(t)-\delta(t-t_0)]}$ impulsetan türevi

since $x(t)*\delta(t-\Delta)=x(t-\Delta)$

$x(t)*\frac{d\delta_\Delta(t)}{dt}$ bakalım türevi ile $x(t)$ conv. edersek ne olur?

$=\frac{x(t)*\delta(t)-x(t)*\delta(t-\Delta)}{\Delta}$

$=\lim_{\Delta\to 0}\frac{x(t)-x(t-\Delta)}{\Delta}$ //

$\boxed{x(t)*\frac{\delta(t)}{dt}=\frac{x(t)}{dt}}$

$\delta(t)$ ye neyi yaparsak impulse o işlemi conv. ettiğimiz zaman

$$\frac{d\delta(t)}{dt} = u_1(t) \qquad x(t)*u_1(t) = \frac{dx(t)}{dt}$$

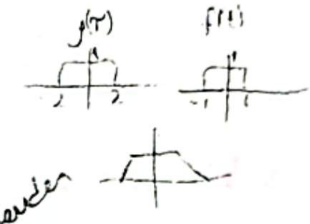
$$\frac{d^2\delta(t)}{dt^2} = u_2(t) \qquad x(t)*u_2(t) = \frac{d^2x(t)}{dt^2}$$

$$\boxed{u_1(t)*u_1(t) = \frac{d^2\delta(t)}{dt^2}}$$

$$x(t)*u_2(t) = [x(t)*u_1(t)]*u_1(t) = \frac{d\,x(t)}{dt}*u_1(t) = \frac{d^2x(t)}{dt^2}$$

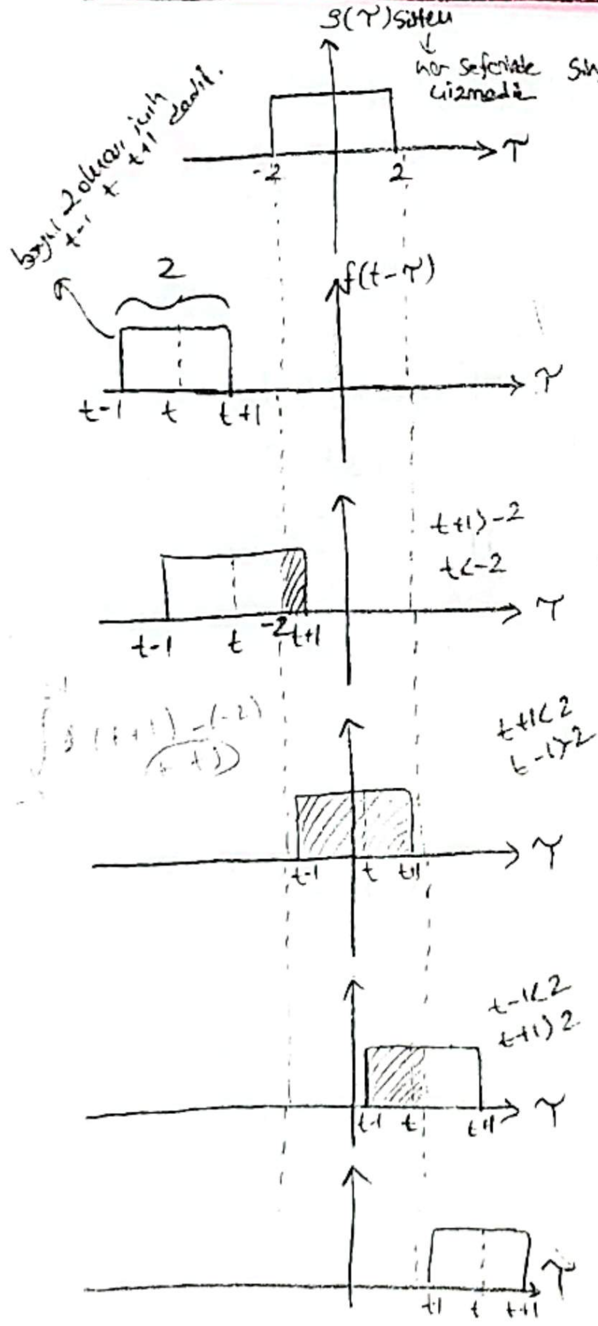
$$\frac{d^2\delta(t)}{dt^2} \xrightarrow{int.} \frac{d\delta(t)}{dt} \xrightarrow{int.} \delta(t) \xrightarrow{int.} u(t) \xrightarrow{integration} \int_{-\infty}^{t} u(t)dt$$

$$\frac{d^2\delta(t)}{dt^2} \xleftarrow{deriv.} \frac{d\delta(t)}{dt} \xleftarrow{deriv.} \delta(t) \xleftarrow{deriv.} u(t) \xleftarrow{derivative} \int_{-\infty}^{t} u(t)dt$$



Convolution on Continious Time Signals

her seferinde sinyal sistemler geçiyor çizmede



sinyalini g(τ) sistemden geçiriyoruz

$t < -3$
$g(\tau)\, f(t-\tau) = 0$

$-3 \le t < -2$
$\int g(\tau)f(t-\tau)\,d\tau = \underbrace{(\overset{\text{eni}}{t+1-(-2)}) \cdot \overset{\text{boyu}}{1}}_{\text{taralı alan}} = t+3$

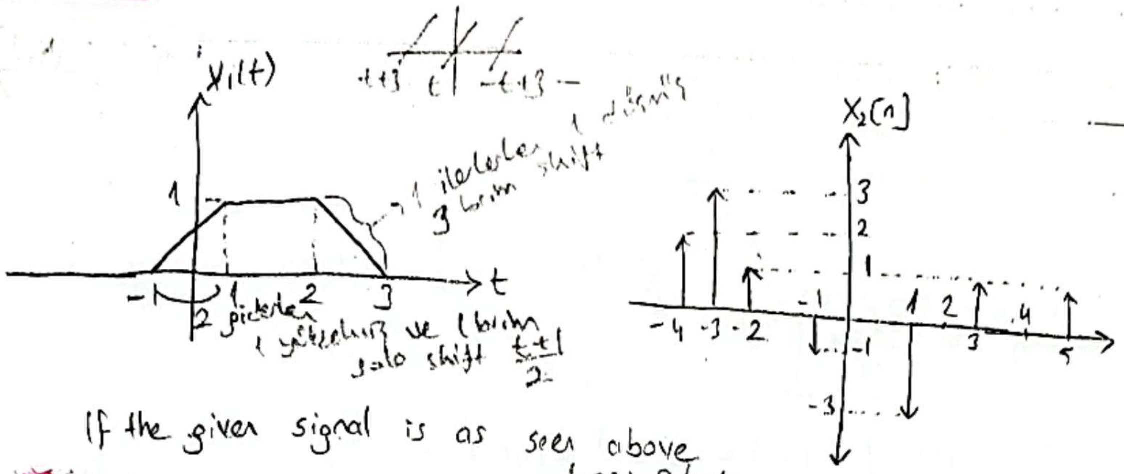
$-1 \le t < 1$
$(t+1)-(t-1)$
$\int g(\tau)f(t-\tau)\,dt = 2 \cdot 1 = 2$

$1 \le t < 3$
$\int g(\tau)f(t-\tau) = (2-(t-1)) \cdot 1 = 3-t$

$3 \le t < \infty$
$\int g(\tau)f(t-\tau)\,d\tau = 0$

$$y(t) = \begin{cases} 0 & t < -3 \\ t+3 & -3 \le t < -2 \\ 2 & -1 \le t < 1 \\ 3-t & 1 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$

Example:



If the given signal is as seen above

a) Evaluate the graph of $-2x_1(3-\frac{1}{2}t)+1$ → en son 1 birim yukarı it.

boyu 2 katı

2 ile çarp t eksenini

x eks. göre simetri

y eks. göre simetri

önce parantez içini yap. shift önce yapılır.

b) Evaluate $-2x_2[2-2n]$

2) 2 ye böl (x eks.) tam sayı olmayanları sil.

1) 2 birim sola kaydır

3) x eks. göre simetrisini al

4) boyu 2 katına çıkar.

5) Y eksenine göre simetrisini al

c) Calculate the power and energy of $x_1(t)$.

$$x_1(t)=\begin{cases}\frac{t+1}{2}, & -1\le t<1\\ 1, & -1\le t\le 2\\ -t+3, & 2\le t\le 3\\ 0, & \text{else}\end{cases}\qquad |x_1(t)|^2=\begin{cases}\left|\frac{t+1}{2}\right|^2, & -1\le t<1\\ 1, & 1\le t<2\\ |-t+3|^2, & 2\le t<3\\ 0, & \text{else}\end{cases}$$

$\boxed{\text{Energy of } x_1(t):}$ $$E_x=\lim_{T\to\infty}\int_{-T/2}^{T/2}|x_1(t)|^2dt$$

$$E_x=\int_{-1}^{1}\frac{(t+1)^2}{4}dt+\int_{1}^{2}1\,dt+\int_{2}^{3}(-t+3)^2dt=2{,}5$$

$\boxed{\text{Power} =}$ $$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x_1(t)|\,dt=\lim_{T\to\infty}\frac{1}{T}\{E_x\}=\lim_{T\to\infty}\frac{2{,}5}{T}=\frac{2{,}5}{\infty}=0$$

sinyal periyodik olsaydı periyoda bölerdik!

Disc	Cont
$\tilde{F}$	$\tilde{F}$
$-$	$\mathcal{L}$
Z	$-$

Fourier Analysis for cont. time Signals and Systems

Periodic signals

$x(t)$ is periodic signal with period T

if $x(t) = x(t+kT)$ $k \in \mathbb{Z}$ (integer) $T \in \mathbb{R}$ (real)

Ex $x(t) = \cos(200\pi t)$ what is the period of the signal?

$x(t) = x(t+T)$

$\cos(200\pi t) = \cos(200\pi(t+T))$

$\cos(200\pi t + 2\pi) = \cos(200\pi t + 200\pi T) \Rightarrow 200\pi T = 2\pi$ $\quad T = \frac{1}{100}$

Let $y(t) = x_1(t) + x_2(t)$

Let period of $x_1(t)$ is T_1

" " " $x_2(t)$ is T_2

Then the period of their sum $(y(t))$ will be evaluated by LCM (Least common multiple) of these values as $LCM(T_1, T_2)$

Example

$f(t) = \cos\left(\frac{\pi}{2}t\right) + \cos\left(\frac{\pi}{7}t\right)$ period of $f(t)$

$\cos\left(\frac{\pi}{2}t\right) \Rightarrow T = 4$

$\cos\left(\frac{\pi}{7}t\right) \Rightarrow T = 14$

period $= LCM(4, 14)$
$= 28$

Theorem

If $x(t)$ is a periodic signal with period T such that $x(t+kT)$ then $x(t)$ signal can be written in terms of sines and cosines.

$$x(t) = A[0] + \sum_{k=1}^{\infty} A[k]\cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} B[k]\sin\left(\frac{2\pi k t}{T}\right)$$

$$A[0] = \frac{1}{T}\int_T x(t)\,dt \qquad A[k] = \frac{2}{T}\int_T x(t)\cos\left(\frac{2\pi k t}{T}\right)dt$$

$$B[k] = \frac{2}{T}\int_T x(t)\sin\left(\frac{2k\pi t}{T}\right)dt$$

Given $a[k], b[k], A[0]$ are all real numbers. There can also be together and we can obtain complex Fourier series representation of periodic signals as

$$x(t) = \sum_{k=-\infty}^{\infty} x(k)\, e^{\frac{jk2\pi t}{T}} \qquad x[k] = \frac{1}{T}\int_T x(t)\, e^{\frac{-jk2\pi t}{T}}\, dt$$

where $x(k)$ is a complex number and real coefficients can be obtained by using $x(k)$

$$A[k] = [x(k) + x^*(k)].\,2 \qquad k \neq 0$$

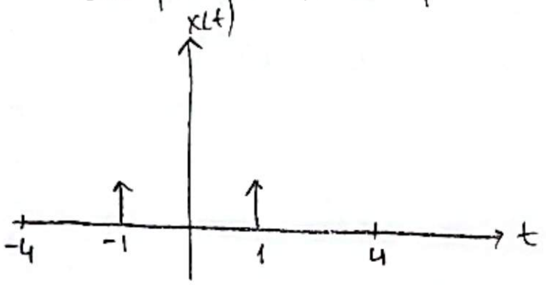
$$B[k] = [x(k) - x^*(k) j\,]$$

$$A(0) = x[0] \quad \text{or} \quad x[k] = \frac{1}{2}\left(A(k) - jB(k)\right)$$

$$A(0) = x(0)$$

Example

One period of a periodic signal is given below



$x(t)$ has period of 8 (given)

Find fourier series representation we can either use

I $\quad x[t] = A[0] + \sum_{k=1}^{\infty} A[k] \cos\left(\frac{k2\pi}{T}t\right) + \sum_{k=1}^{\infty} B[k] \sin\left(\frac{k2\pi t}{T}\right)$

II or $\quad x(t) = \sum_{k=-\infty}^{\infty} x[k]\, e^{\frac{jk2\pi . t}{T}}$ for the use of first one; we evaluate the coefficients at first : $T = 8$ given

$$A[0] = \frac{1}{8}\int x(t)\,dt = \frac{1}{8}\int_{-4}^{4} \overbrace{\left(\delta(t+1) + \delta(t-1)\right)}^{x(t)} dt = \frac{1}{8}(1+1) = \frac{1}{4}$$

$$A[k] = \frac{2}{8}\int_{-4}^{4} \left(\delta(\overset{t=-1}{t+1}) + \delta(\overset{t=1}{t-1})\right) \cos\left(\frac{k2\pi t}{T}\right) dt$$

→ -1 ve 1 -4 ve 4'ün arasında olmasa alırdık 0 olurdu.

↳ Sadece t yerine (1) de değer ald. için integrali almadan 1 dedik.

$$\cos(-\theta) = \cos(\theta)$$

$$A[k] = \frac{2}{8}\left[\cos\left(\frac{-2k\pi}{T}\right) + \cos\left(\frac{2k\pi}{T}\right)\right] = \frac{1}{2}\cos\left(\frac{2k\pi}{T}\right)\Big] = \frac{1}{2}\cos\left(\frac{k2\pi}{T}\right)$$

$$\sin(-\theta) = -\sin\theta$$

$$B[k] = \frac{2}{8}\int_{-4}^{4} \left[\left(\delta(t+1) + \delta(t-1)\right) . \sin\left(\frac{k2\pi t}{T}\right)\right] dt = \frac{1}{4}\left[\sin\left(\frac{k2\pi(-1)}{T}\right) + \sin\left(\frac{k2\pi(1)}{T}\right)\right]$$

$$= 0 = \varnothing \text{ boş küme}$$

Now these coefficients can be used to evaluate the Fourier series Representation

$$x(t) = \sum_{-\infty}^{\infty} x[k]\, e^{jk\frac{2\pi t}{T}} \rightarrow \text{Finding } x(t) \text{ using } x[k]$$

$$x[k] = \sum_{-\infty}^{\infty} x[t] . e^{-jk\frac{2\pi t}{T}} \rightarrow \text{Finding } x[k] \text{ using } x(t)$$

II) $$x[k] = \frac{1}{8} \int_{-4}^{4} \overbrace{(\delta(t+1)+\delta(t-1))}^{x(t)} . e^{\frac{-jk2\pi}{T} . t}\, dt$$

devamı

$$\boxed{x(t) = \sum_{-\infty}^{\infty} \overbrace{\frac{1}{4}\cos\left(\frac{2\pi k}{T}\right)}^{x[k]} e^{jk\frac{2\pi t}{T}}}$$

$$x[k] = \frac{1}{8}\left[e^{\frac{-jk2\pi(-1)}{T}} + e^{\frac{-jk2\pi(1)}{T}}\right] = \frac{1}{8}\left[e^{\frac{jk2\pi}{T}} + e^{\frac{-jk2\pi}{T}}\right]$$

$$\boxed{x[k] = \frac{1}{4}\cos\left(\frac{2\pi k}{T}\right)}$$

buna eşit $\Rightarrow$ $$x(t) = A[0] + \sum_{k=1}^{\infty} A[k] \cos\left(\frac{k2\pi t}{T}\right) + \sum_{k=1}^{\infty} B[k] \sin\left(\frac{2k\pi t}{T}\right)$$

Summary

Let $x(t)$ be a periodic signal $x(t) = x(t+\tau)$

Fourier series representation of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{jk\frac{2\pi t}{T}} \quad \text{and} \quad x(k) = c_1 c_2 \int_T x(t)\, e^{-jk\frac{2\pi t}{T}}\, dt$$

dışarıda ... içerde

$$c_1 = \frac{1}{\sqrt{T}}, \quad c_2 = \frac{1}{\sqrt{T}}$$

$$c_1 = 1, \quad c_2 = \frac{1}{T}$$

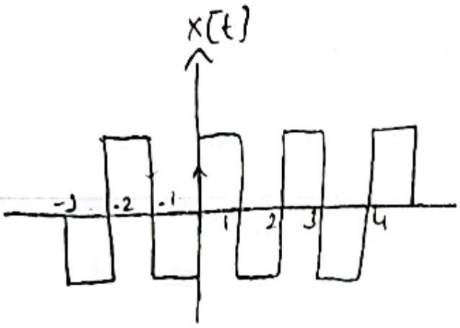
$$c_1 = \frac{1}{T}, \quad c_2 = 1$$

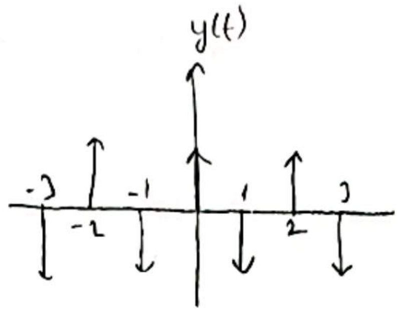
$$\left.\right\} c_1 . c_2 = \frac{1}{T}$$

Property

$x(t) \rightarrow$ periodic $\quad x(t) = x(t+T)$ if $y(t) = \frac{dx(t)}{dt}$

then $y(t)$ has to be periodic





$y(t) = y(t+T)$

in determining the Fourier Series coefficients of it

$$x(t) = \sum_{k=-\infty}^{\infty} x[k].e^{\frac{jk2\pi t}{T}} \quad \text{where} \quad w_0 = \frac{2\pi}{T} \qquad y(t) = \frac{dx(t)}{dt}$$

$$y(t) = \sum_{k=-\infty}^{\infty} x[k]\, jkw_0 . e^{\frac{jk2\pi t}{T}}$$

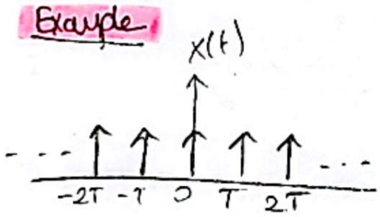
$$\Rightarrow y[k] = x[k]\, jk\, w_0$$

in general if $y(t) = \frac{d^n x(t)}{dt^n}$ then

$$y[k] = (jw_0 k)^n x[k]$$

This property will also be used for determining the fourier series coefficients of periodic signals.

Example



$$x(t) = \sum_{k=-\infty}^{\infty} (\delta(t-kT))$$

evaluate FSR of $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} x(k)\, e^{jkw_0 t} \qquad w_0 = \frac{2\pi}{T}$$

$$x[k] = \frac{1}{T}\int x(t)\, e^{-jkw_0 t}\, dt$$

$$x[k] = \frac{1}{T}\int_{-T/2}^{T/2} \underbrace{\delta(t)\, e^{-jkw_0 t}}_{t=0}\, dt$$

$$x[k] = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{0} = \frac{1}{T}.1 = \frac{1}{T}$$

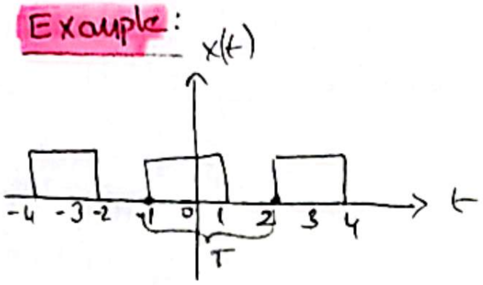
$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T}}_{x[k]} e^{jkw_0 t}$$

$$= \boxed{\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jkw_0 t}}$$

Note that,

$$\int f(t)\,\delta(t-t_0)\,dt = \int f(t_0)\,\delta(t_0-t_0)\,dt = f(t_0)\int \delta(t_0-t_0)\,dt$$
$$= f(t_0).1 = f(t_0)$$

Example:



$T=3$ for given periodic signal evaluate F.S.R $T=3$ for

$x(t) = x(t+T)$

we may use the derivative property for $x(t) \Rightarrow y(t) = \frac{dx(t)}{dt}$

$y(k) = -jk\omega_0\, x(k)$

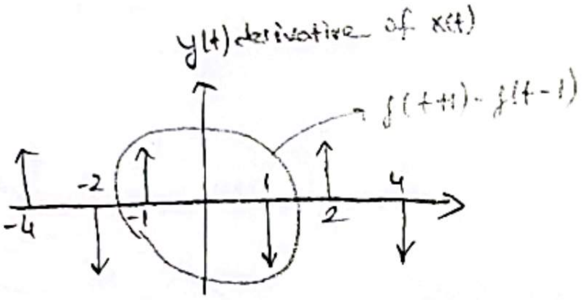
$x(k) = -\frac{1}{jk\omega_0}\, y(k)$

∇ $x[k] = \sum_{k=-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}$ $\omega_0 = \frac{2\pi}{T}$

$\frac{d\,x[k]}{dt} = -jk\omega_0\, x(t) e^{-jk\omega_0 t}$

$\underbrace{\frac{d\,x[k]}{dt}}_{y[k]} = \underbrace{\sum_{k=-\infty}^{\infty} \underbrace{x(t)\, e^{-jk\omega_0 t}\, jk\omega_0}_{y(t)}}_{x[k]\cdot}$

$\boxed{x[k] = \frac{y[k]}{-jk\omega_0}}$



$y(k) = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$

$\boxed{\text{for } T=3}$ $y(k) = \frac{1}{3}\int_{-3/2}^{3/2} \underbrace{\delta(t+1) - \delta(t-1)}_{y(t)}\, e^{-jk\omega_0 t}\, dt$

$y(k) = \frac{1}{3}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) = \frac{1}{3} 2 j \sin(\omega_0 t)$ where $\omega_0 = 2\pi/3$

$2\pi f = \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$x(k) = \frac{1}{jk\omega_0}\, y(k)$

$\Rightarrow x[k] = -\frac{1}{jk\, 2\pi/3} \cdot \frac{1}{3} 2j \sin\left(\frac{2\pi t}{T}\right)$

$x[k] = -\frac{1}{k\pi} \sin\left(\frac{2\pi t}{3}\right)$ from here $x(t)$ can be evaluated using

$x(t) = \sum_{k=-\infty}^{\infty} x[k] . e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} -\frac{1}{k\pi} \sin\left(\frac{2\pi t}{3}\right) e^{jk\left(\frac{2\pi t}{3}\right)}$ (ω_0)

* Fonksiyonumuz delta ise direk onu alıp çözüyoruz. unit step ise türev alıp deltaya benzetiyoruz. Ramp ise 2 kere türev alıyoruz

$x(t) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-jk\omega_0}$

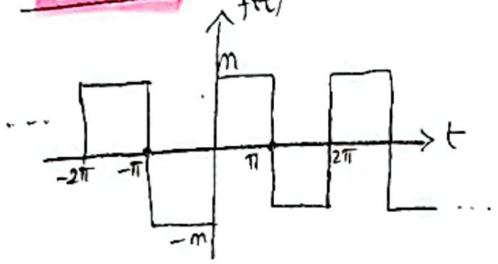
$x(t) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-jk\omega_0 t}$

$$\underbrace{\frac{dx(t)}{dt}}_{y[k]} = \sum_{k=-\infty}^{\infty} \underbrace{x(t)\, e^{-jk\omega_0 t}}_{x[k]} \cdot -jk\omega_0$$

$$\boxed{y[k] = x[k] \cdot -jk\omega_0}$$

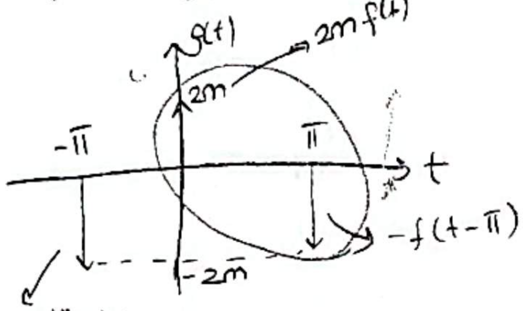
$e^{-jk\pi} = \cos(-k\pi) + j\cancel{\sin(-k\pi)}$
$e^{-jk\pi} = \cos(-k\pi) = \cos(k\pi)$

Example



period of $f(t) \Rightarrow 2\pi$
Evaluate FSR

$$g(t) = \frac{d f(t)}{dt}$$



periyodik bir fonk. old. için bir periyodunu aldık.

$$G(k) = \frac{1}{T} \int_T g(t)\, e^{-jk\omega_0 t}\, dt$$ 0 ile 2π arası dalga

$$= \frac{1}{2\pi} \int_0^{2\pi} 2m\left[\delta(t) - \delta(t-\pi)\right] e^{-jk\omega_0 t}\, dt \qquad t=0 \quad t=\pi$$

$$= \frac{m}{\pi}\left(1 - e^{-jk\omega_0 \pi}\right)$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$G[k] = \frac{m}{\pi}\left(1 - e^{-jk\pi}\right) = \frac{m}{\pi}\left(1 - \cos(k\pi)\right)$$

since $g(t)$ is defined as $g(t) = f'(t)$

$$\longrightarrow G[k] = j\omega_0 k\, f[k]$$

$$G[k] = jk \frac{2\pi}{2\pi} f[k] \Rightarrow f[k] = \frac{G[k]}{jk} = \frac{m}{jk\pi}\left(1 - \cos(k\pi)\right) \quad (j)$$

$$f[k] = \frac{jm}{k\pi}\left(\cos(k\pi) - 1\right)$$

$$\boxed{\cos(k\pi) = (-1)^k}$$

$$f(k) = \frac{jm}{k\pi}\left((-1)^k - 1\right)$$

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\, e^{jk\omega t}\, dt \qquad f(t) = \sum_{k=-\infty}^{\infty} \frac{jm}{k\pi}\left((-1)^k - 1\right) e^{jkt} \qquad \omega_0 = \frac{2\pi}{2\pi}$$

Example

a) $x(t) = \sin\left(\overbrace{\frac{2\pi}{T}.t}^{\theta}\right)$ $T=2\pi$ Find the fourier series coefficients of $x(t)$

$$X[k] = \frac{1}{T}\int_T x(t)\, e^{-jk\frac{2\pi}{T}.t}\, dt$$

$$\boxed{\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)}$$

buraya yaz

$$x(t) = \frac{1}{2j}\left[e^{jk\frac{2\pi}{T}.t} - e^{-j\left(\frac{2\pi t}{T}\right)}\right]$$

and

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\frac{2\pi t}{T}} \qquad x(t) = \overset{-\infty}{\cdots}\ x[-1].e^{-j\frac{2\pi t}{T}} + x[0] + x[1]e^{\frac{j2\pi t}{T}} \overset{\infty}{\cdots}$$

$$\text{So } X[k] = \begin{cases} -\frac{1}{2j} & k=-1 \\ \frac{1}{2j} & k=1 \\ 0 & \text{else} \end{cases}$$

b) $\sin\theta = \frac{1}{2j}\left[\underset{k=1}{e^{j\pi t}} - \underset{k=-1}{e^{-j\pi t}}\right]$

$x(k) = k=1 \rightarrow 1/2j$

$k=-1 \rightarrow -\frac{1}{2j}$

Example

$$x[t] = \underset{T=1}{\cos(\overbrace{2\pi t}^{2\pi f t})} + \underset{T=1/2}{\cos(4\pi t)} + \underset{T=1/3}{\cos(6\pi t)} \qquad LCM\left(1, \tfrac{1}{2}, \tfrac{1}{3}\right) = 1$$

$$x(t) = \frac{1}{2}\left[\underset{k=1}{e^{j2\pi t}} + \underset{k=-1}{e^{-j2\pi t}} + \underset{k=2}{e^{j4\pi t}} + \underset{k=-2}{e^{-j4\pi t}} + \underset{k=3}{e^{j6\pi t}} + \underset{k=-3}{e^{-j6\pi t}}\right]$$

$jk2\pi t = j6\pi t$

for $k = 1, -1, 2, -2, 3, -3$ $\quad 1/2$

else $\quad 0$

Properties of Fourier Series Coefficients

$x(t) \rightarrow$ periodic with period T

$x(k) \rightarrow$ Fourier Series coefficients of $x(t)$

$y(t) \rightarrow$ periodic with period T

$y(k) \rightarrow$ Fourier series coefficients of $y(t)$

	Signal		FCS (Fourier series Coefficients)
1)	$A\,x(t) + B\,y(t)$	$\xleftrightarrow{FSC}$	$A\,x(k) + B\,y(k)$
2)	$x(t - t_0)$	$\xleftrightarrow{FSC}$	$x(k).e^{-jk\frac{2\pi}{T}.t_0}$
3)	$e^{jm\frac{2\pi t}{T}}.x(t)$	$\xleftrightarrow{FSC}$	$x(k-m)$
4)	$x^*(t)$	$\xleftrightarrow{FSC}$	$x^*(-k)$

5) $\int_T x(\tau)y(t-\tau)d\tau \xleftrightarrow{FSC} T.x(k)y(k)$

6) $x(-t) \xleftrightarrow{FSC} x[-k]$

7) $x(t)y(t) \xleftrightarrow{FSC} x[k] * y[k] = \sum_{k=-\infty}^{\infty} x(l]\, y(k-l)$

8) $\frac{d^n x[n]}{dt^n} \xleftrightarrow{FSC} (jk\omega_0)^n x[k]$

9) $\int_{-\infty}^{t} x(t)dt \xleftrightarrow{FSC} \frac{1}{jk\omega_0} x[k]$

10) $\int_T |x(t)|^2 dt \xleftrightarrow{FSC} T.\sum_{k=-\infty}^{\infty} |x(k)|^2$

Proof of 2

$$\left.\begin{aligned} x(t) &\longleftrightarrow x[k] \\ y(t) &= x(t-t_0) \end{aligned}\right\} y(k) = e^{-jk\frac{2\pi}{T}t_0} x[k]$$

$$y[k] = \frac{1}{T}\int_T y(t)\, e^{-jk\frac{2\pi t}{T}} dt$$

$$= \frac{1}{T}\int_T x(t-t_0).e^{-jk\frac{2\pi t}{T}} dt \qquad t' = t - t_0 \Rightarrow t = t' + t_0$$

$$= \frac{1}{T}\int_T x(t')\, e^{-jk\frac{2\pi(t'+t_0)}{T}} dt = \frac{1}{T}\int_T x(t') e^{-jk\frac{2\pi t}{T}.t_0} . e^{jk\frac{2\pi}{T}.t'} dt$$

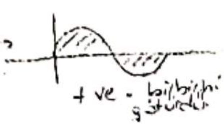
$$= e^{-jk\frac{2\pi}{T}t_0} . \underbrace{\frac{1}{T}\int_T x(t').e^{-jk\frac{2\pi}{T}t'} dt'}_{x[k]} \Rightarrow y[k] = e^{-jk\frac{2\pi}{T}t_0} . x[k]$$

Example

$$\int_T e^{j(k-m)\frac{2\pi t}{T}} dt = ?$$

$$\int_T e^{j\overbrace{(k-m)\frac{2\pi t}{T}}^{a}} dt = \int_T \cos\Big(\overbrace{(k-m)\frac{2\pi}{T}.t}^{a}\Big) + j\sin\Big(\overbrace{(k-m)\left(\frac{2\pi t}{T}\right)}^{a}\Big) dt$$

if $k \neq m \Rightarrow \int_T e^{j(k-m)\frac{2\pi}{T}.t} dt = \begin{cases} T & k = m \\ 0 & k \neq m \end{cases}$ →



Proof of $\circledast$ ~~T~~

$$\underbrace{\int_T x(\tau)y(t-\tau)d\tau}_{\text{periodic convolution}} \xleftrightarrow{FSC} T.\, x[k]\, y[k]$$

$$z(t) = x(t) * y(t) = \int x(\tau)y(t-\tau)d\tau$$

$$x(\tau) = \sum_k x(k)\, e^{jk\frac{2\pi}{T}\tau}$$

$$y(t) = \sum_m y(m)\, e^{jm\frac{2\pi t}{T}} \qquad y(t-\tau) = \sum_m y(m)\, e^{jm\frac{2\pi}{T}(t-\tau)}$$

$$z(t) = \int_T \underbrace{\sum_k x(k)\, e^{jk\frac{2\pi}{T}\tau}}_{x(\tau)} \underbrace{\sum_m y(m)\, e^{jm\frac{2\pi}{T}(t-\tau)}}_{y(t-\tau)} d\tau$$

$$z(t) = \sum_k \sum_m x(k)y(m) \underbrace{\int_T e^{j(k-m)\frac{2\pi}{T}.\tau} d\tau}_{\substack{T \text{ if } k=m \\ 0 \text{ else}}} . e^{jm\frac{2\pi t}{T}}$$

$$z(t) = \sum_k \underbrace{x(k)y(k)T}_{z(k)}\, e^{jk\frac{2\pi t}{T}} \Rightarrow \underbrace{x(t) * y(t)}_{z(t)} \overset{FSC}{\Longrightarrow} \underbrace{T.x(k)y(k)}_{z(k)}$$

The Continuous Time Fourier Transform (F.T)

$\tilde{x}(t) \rightarrow$ periodic signal

$x(t) \rightarrow$ one period of the periodic $\tilde{x}(t)$ signal.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT)$$

$$\tilde{x}(t) \xleftrightarrow{FSC} \tilde{x}[k]$$

$$\tilde{x}[k] = \frac{1}{T}\int_{T} \tilde{x}(t).\, e^{\frac{-jk2\pi t}{T}} = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{\frac{-jk2\pi t}{T}} dt$$

$\hookrightarrow$ one period of $x(t)$

$$T.\, \tilde{x}[k] = \int_{-\infty}^{\infty} x(t)\, e^{\frac{-jk2\pi t}{T}} dt \qquad \boxed{X(w) = \int_{-\infty}^{\infty} x(t) e^{-jkw_0 t} dt} \quad w = kw_0$$

F.T. of aperiodic signal $x(t)$

on the other hand;

$X(w) = T.\tilde{x}[k] \qquad w = k.w_0$

$\tilde{x}[k] = \frac{X(kw_0)}{T_{\cdot 2\pi}} \qquad \tilde{x}(t) = \sum_k \frac{w_0}{2\pi} X(kw_0) e^{jkw_0 t}$

$\frac{1}{T}$

for $T \Rightarrow \infty \quad \tilde{x}(t) = x(t)$, $\quad w = \frac{2\pi}{T}$

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw}$$ inverse F.T of periodic signal

Summary

$x(t) \longrightarrow$ periodic

Fourier transform of $x(t)$

$$X(w) = \int_{-\infty}^{\infty} x(t).e^{-jwt}\, dt$$

Inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w) e^{jwt}\, dt$$

Note:

if $X(w) = \overset{C_1}{\frac{1}{\sqrt{2\pi}}}\int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt \qquad x(t) = \overset{C_2}{\frac{1}{\sqrt{2\pi}}}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dt$

$C_1.C_2 = \frac{1}{2\pi}$ if $C_1 = 1 \quad C_2 = \frac{1}{2\pi}$

$C_2 = 1 \quad C_1 = \frac{1}{2\pi}$, if $C_1 = \frac{1}{\sqrt{2\pi}} \quad C_2 = \frac{1}{\sqrt{2\pi}}$

Convergence of F.T.

Every signal may not have F.T. in general $\int_{-\infty}^{\infty} x(t).e^{-jwt}\, dt$

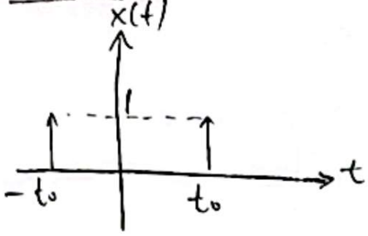
$$\left| \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt \right| < \infty$$

$$\underbrace{\left| \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt \right|}_{\text{finite value}} \leq \int_{-\infty}^{\infty} |x(t)|\, e^{-jwt}\, dt < \infty$$

Note: $|e^{j\alpha}| = 1$

Hence $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \Rightarrow$ Condition for convergence of F.T. If a signal is not absolutely integrable then F.T doesn't exist.

Example



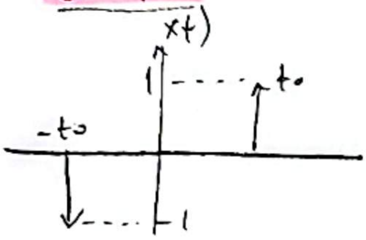
$x(k) \to$ fourier series coefficient?
$X(\omega) = ? \to$ fourier transform?

$$x(t) = \delta(t-t_0) + \delta(t+t_0)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} [\delta(t-t_0) + \delta(t+t_0)]\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t_0} + e^{j\omega t_0} = 2\cos(\omega t_0)$$

Example



$X(\omega) = ?$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [\delta(t-t_0) - \delta(t+t_0)]\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t_0} - e^{j\omega t_0} = -2j\sin(\omega t_0)$$

Example

$x(t) \longrightarrow$ periodic with T
$y(t) \longrightarrow$ periodic with T

Show that

$$\boxed{x(t)\, y(t) \xleftrightarrow{FSC} x(k) * y(k)}$$

$$z(t) = x(t)\, y(t)$$

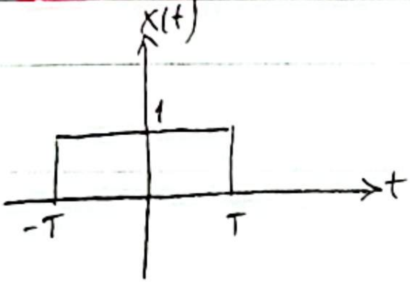
$$z(k) = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int x(t)\, y(t)\, e^{-jk\omega_0 t}\, dt$$

$$x(t) \to \sum_m x(m)\, e^{jm\frac{2\pi}{T}t}\, dt$$

$$= \sum_m x(m) \underbrace{\frac{1}{T}\int_T y(t)\, e^{-j(k-m)\frac{2\pi}{T}t}\, dt}_{y(k-m)}$$

$$= \sum_m x(m)\, y(k-m) = x(k) * y(k)$$

Example



$x(w) = ?$

$$x(w) = \int_{-\infty}^{\infty} x(t) \cdot e^{-jwt} \, dt$$

$$= \int_{-T}^{T} 1 \cdot e^{-jwt} \, dt = -\frac{1}{jw} \cdot e^{-jwt} \Big|_{-T}^{T}$$

$$= -\frac{1}{jw} \left(e^{-jwT} - e^{+jwT} \right) = 2j \left(\frac{1}{jw} \sin(wT) \right) = \frac{2}{w} \sin(wT)$$

yerine yazaldı

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(w) e^{jwt} \, dw$$

$$1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{w} \sin(wT) e^{jwt} \, dw$$

$$\pi = \int_{-\infty}^{\infty} \frac{\sin(wT)}{w} \cdot e^{jwt} \, dw$$

Relation Between F.S.C of periodic Signal

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t - mT)$$

$\tilde{x}(t) \longrightarrow$ periodic with period T
$x(t) \longrightarrow$ one period of $\tilde{x}(t)$

$$x(t) = \begin{cases} \tilde{x}(t) & 0 \leq t < T \\ 0 & \text{else} \end{cases}$$

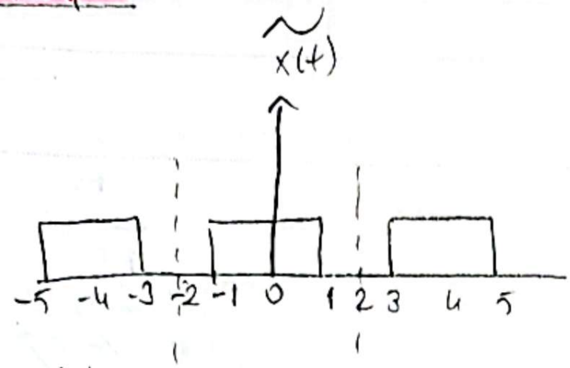
$$\tilde{x}(t) \xleftrightarrow{FSC} \tilde{x}(k)$$

$$\tilde{x}(t) \xrightarrow{FT} x(w)$$

$$\tilde{x}(t) = \sum_{k} x(t - kT) \qquad \tilde{x}(t) = \tilde{x}(t+T)$$

$$\tilde{x}(k) = \frac{1}{T} x(w) \Big|_{w = kw_0} , \quad w_0 = \frac{2\pi}{T}$$

Example



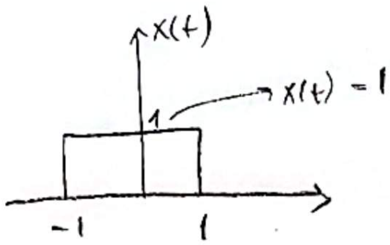
$T = 4$ sec.

$\tilde{x}(t) \rightarrow$ periodic

$\tilde{x}(t) \rightarrow \tilde{x}(t + \underset{\hookrightarrow T}{4})$

$x[k] = ?$ (fourier series coefficient)

since $x(t)$ is a single period of $\tilde{x}(t)$
It can be drawn as



periyodik bir fonksiyonun yalnız bir periyodunu alırız.

$$X(w) = \int_{-1}^{1} x(t) e^{-jwt} dt = \int_{-1}^{1} 1 \, e^{-jwt} dt = \frac{1}{-jw}\left(e^{-jw} - e^{jw}\right)$$

$$= \frac{2}{w} \sin(w)$$

$$* \sin w = \frac{e^{jw} - e^{-jw}}{2j}$$

* Fourier transformunu $\frac{1}{T}$ ile çarp, w yerine kw_0 yazarsak F.S.C bulmuş oluruz.

If the Fourier series Coefficients are desired to be evaluated

$$\left(\tilde{x}(t) \xrightarrow{FSC} \tilde{x}[k]\right)$$

$$\boxed{\underbrace{\tilde{x}[k]}_{F.S.C} = \frac{1}{T} \underbrace{X(w)}_{F.T}\Big|_{w = kw_0}}$$

$w_0 = \frac{2\pi}{T}$ where T is given as 4.

$$x[k] = \frac{1}{4}\,\frac{2}{w}\,.\sin w \,\Big|_{w = k.\frac{2\pi}{\underset{T}{4}} = \frac{k\pi}{2}}$$

$$\tilde{x[k]} = \frac{1}{4}\,\frac{2}{\frac{k\pi}{2}}\,\sin\left(\frac{k\pi}{2}\right) \qquad \boxed{\tilde{x[k]} = \frac{1}{k\pi}\,\sin\left(\frac{k\pi}{2}\right)}$$

$$\begin{tikzcd} x(t) \arrow[r, shift left, "FT"] \arrow[d, shift right, "FSC"'] & X(w) \arrow[l, shift left, "IFT"] \arrow[dl, shift left, "/T"] \\ x(k) \arrow[u, shift right] \arrow[ur, shift left, "xT"] & \end{tikzcd}$$

?

$$\tilde{x(t)} = \sum_{k} \tilde{x[k]}\, e^{jk\frac{2\pi}{4}t}$$

$$x(t) = \sum_{k} \frac{1}{k\pi}\,\sin\frac{k\pi}{2}\, e^{jk\frac{\pi}{2}t}$$

x(t) is re-evaluated from its F.S.C

$$\begin{tikzcd} x(t) \arrow[r] \arrow[dr] & X(w) \\ & x[k] \end{tikzcd} \qquad \tilde{x[k]} = \frac{1}{T}\,X(w)$$

$$\begin{tikzcd} x(t) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\text{re evaluated}"] & X(w) \arrow[dl] \\ x(k) & \end{tikzcd} \qquad x[k] = \frac{1}{T}\,X(w)\Big|_{w = \frac{2\pi}{T}.k}$$

ex

$$x(n) = \left[-1 + \cancel{u(n-1)} + \sum_{k=-2}^{1} k\,\delta(n-k) + 2r[3n] - 4r(n-5) + 2u(n-6) - \cancel{u(n-1)}\right]$$

axis	-2	-1	0	1	2	3	4	5	6	
	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
	-2	0	0	0	0	0	0	0	0	$-2\delta(n+2)$
	0	-1	0	0	0	0	0	0	0	$-\delta(n+1)$
	0	0	0	1	0	0	0	0	0	$\delta(n-1)$
	0	0	0	6	12	18	24	30	36	$r(6n)$
	0	0	0	0	0	0	0	0	-4	$-4r(n-5)$
	0	0	0	0	0	0	0	0	2	$2u(n-6)$

* $X(w)$'dan $x(t)$'yi bulurken $\frac{1}{2\pi}$ çarpanı geliyor!

Example

$X(w) = 2\pi\,\delta(w-w_0)$
$x(t) = ?$

* fourier transformunu vermiş sinyali soruyor.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\,e^{jwt}\,dw = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(w-w_0)\,e^{jwt}\,dw \quad w=w_0$$

$$x(t) = e^{jw_0 t}$$

$$\boxed{e^{jw_0 t k} \xleftrightarrow{F.T} 2\pi\,\delta(w-kw_0)}$$ General formula!

shown that inverse fourier transform of $X(w) = 2\pi\,\delta(w-w_0)$

* $x(t) \xrightarrow{FT} X(w)$, $x(t) \xrightarrow{FSC} X(k)$ $\Big\}_{\times T}$ bulabiliriz.

F.T of periodic Signal

$x(t) \longrightarrow$ periodic with period T

$$x(t) \longrightarrow \sum_k X[k]\,e^{jkw_0 t} \xrightarrow{F.T} X(w) = \sum_k X[k]\,2\pi\,\delta[w-kw_0]$$

fourier transformu

Thus $X(w) = 2\pi \sum_k X[k]\,\delta[w-kw_0]$

$$x(t) = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \underbrace{\cancel{2\pi}\sum_k X[k]\,\delta(w-kw_0)}_{X(w)}\,e^{jwt}\,dw$$

$$x(t) = \sum_k \underbrace{\boxed{X[k]\int_{-\infty}^{\infty}\delta(w-kw_0)}}_{X(w)}\,e^{jwt}\,dw$$

$\boxed{w = kw_0}$ δ old. için sadece bu noktada değer alır.

$$\boxed{x(t) = \sum_k X[k]\,e^{jkw_0 t}}$$

#

Note $\underbrace{e^{jw_0t}}_{x(t)} \xleftrightarrow{F.T} \underbrace{2\pi\delta(w-w_0)}_{x(w)}$

$$x(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

e'nin üzerinde k yok çünkü yanında x[k] yok

$$2\pi\delta(w-w_0) = \int_{-\infty}^{\infty} e^{jw_0t} \cdot e^{-jwt}\, dt$$

$$2\pi\delta(w-w_0) = \int_{-\infty}^{\infty} e^{-j(w-w_0)t}\, dt$$

$$\underbrace{\delta(w-w_0)}_{x(w)} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j(w-w_0)t}\, dt}_{x(t)}$$

$x(w) \rightarrow$ frekans domain

* $w - w_0 = -\Omega$ dersek

$$\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\Omega t} \xrightarrow{F.T} \delta(-\Omega)}$$

$$\begin{array}{ccc} x(t) & \longrightarrow & x(k) \\ \uparrow & \nearrow & \\ x(w) & & \end{array}$$

$$\boxed{\frac{1}{T}\, x(w)\Big|_{w = \frac{k w_0}{T}}}$$

($x(w)$ dan $x(k)$)

bu geçişi yapabilmemiz için T'nin verilmesi lazım

$x(w) \rightarrow x(t) \rightarrow x(k)$

*T yoksa dolanarak bulmamız gerekir.

Properties of F.T

$$x(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

$$x(w) = a(w) + j b(w)$$

where $|x(w)|$ is absolute value of $x(w)$

and $\underline{x(w)}$ is phase " " "

$$|X(\omega)| = \sqrt{a^2(\omega) + b^2(\omega)}$$

$$\angle X(\omega) = \tan^{-1}\left(\frac{b(\omega)}{a(\omega)}\right)$$

Ex

$$y(t) = \frac{d\,x(t)}{dt}$$

$$x(t) \xrightarrow{F.T} X(\omega)$$

$y(t) \longrightarrow ?$ $\quad Y(\omega) = ?$ in terms of $X(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$ → fourier transform formula of $x(t)$

$$y(t) = \frac{d\,x(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,\frac{d(e^{j\omega t})}{dt}\, d\omega$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{(j\omega) X(\omega)}_{Y(\omega)}\, e^{j\omega t}\, d\omega$$

$$\frac{d\,x(t)}{dt} \overset{F.T}{\Rightarrow} Y(\omega) = j\omega\, X(\omega)$$

$$\frac{d^2 x(t)}{dt^2} \overset{F.T}{\Rightarrow} Y^2(\omega) = (j\omega)^2 X(\omega)$$

$$\boxed{\text{In general form} \Rightarrow \frac{d^n x(t)}{dt} \overset{F.T}{\longleftrightarrow} (j\omega)^n X(\omega)}$$

Summary

$x(t)$ → periodic signal

$x(t) = x(t+T)$ → bunun sonucu tam sayı ise periyodik imajiner ise aperiodik.

Fourier Series Representation

Synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} x[k]\, e^{jk\frac{2\pi}{T}t}$$

$$x[k] = \frac{1}{T}\int_{T} x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$k \to$ integer
$T \to$ real number (period)

* If $x(t)$ is a periodic signal F.T of $x(t)$ is

$$X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

Inverse F.T Formula

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw$$

If $x(t)$ is a periodic signal, F.T of $x(t)$ is

$$\boxed{X(w) = \sum_{k=-\infty}^{\infty} 2\pi\, x[k]\, \delta(w - w_0)}$$

Example

$x(t) = \delta(t - t')$ $X(W) = ?$

$$X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt \longleftrightarrow X(w) = \int_{-\infty}^{\infty} \underbrace{\delta(t - t')}_{t=t'} e^{-jwt}\, dt = \underbrace{e^{-jwt'}}_{X(W)}$$

then

$$\boxed{\delta(t - t') \xleftrightarrow{F.T} e^{-jwt'}}$$

time domain

frequency domain

Using inverse F.T

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(w)\, e^{jwt}\, dw$$

$$\xrightarrow{\quad} e^{-jwt'}$$

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jw(t-t')}\, dw$$

$$2\pi\,\delta(t-t') = \int_{-\infty}^{\infty} e^{jw(t-t')}\, dw$$

$\tilde{x}(t) \rightarrow$ periodic func.
$x(t) \rightarrow$ one period of periodic func.

Fourier Analysis for Discrete Time Signals and Systems

$x[n]$ = periodic signal

$x[n] = x[n+N]$

Fourier Series Equations for $x[n]$

$$x[n] = \sum_{k=0}^{N-1} x[k]\, e^{jk\frac{2\pi n}{N}}$$

$$x[k] = \frac{1}{N}\sum_{n,N} x[n]\, e^{-jk\frac{2\pi n}{N}}$$

$\hookrightarrow$ index n, boyu N (period N)

Example

$$\phi[n] = e^{jk\frac{2\pi n}{N}} \qquad k,n \in \mathbb{Z}$$

$$\phi[n] \overset{?}{=} \phi[n+N]$$

$$\phi[n+N] = e^{jk\frac{2\pi}{N}(n+N)}$$

$$= e^{jk\frac{2\pi n}{N}} \cdot e^{jk\frac{2\pi}{N}\cdot N} = e^{jk\frac{2\pi n}{N}} \qquad e^{jk2\pi m} = 1$$

Thus $\phi[n] = \phi[n+N]$

if $n = m.N$
↳ integer value

$$\phi[n] = e^{jk\frac{2\pi m N}{N}} = e^{jk2\pi m} = 1$$

Example

Consider the summation

$$\sum_{n=0}^{N-1} r^n = 1 + r + r^2 + \cdots = \frac{1-r^N}{1-r} \quad \text{if } r \neq 1$$

$$= 1.N \quad \text{if } r = 1$$

Using the given information, calculate

$$\sum_{n=0}^{N-1} \left(e^{jk\frac{2\pi}{N}}\right)^n = ?$$

$mN = k$

if $r=1$: $$\sum_{n=0}^{N-1} \left(e^{jk\frac{2\pi}{N}}\right)^n \longrightarrow \sum_{n=0}^{N-1} \left(e^{jmN\frac{2\pi}{N}}\right)^n$$

$$\longrightarrow \sum_{n=0}^{N-1} \underbrace{\left(e^{jm2\pi}\right)^n}_{1} = \sum_{n=0}^{N-1} 1 = N \quad \text{if } k = 0, \pm N, \pm 2N, \ldots$$

if $k \neq 0$ mN

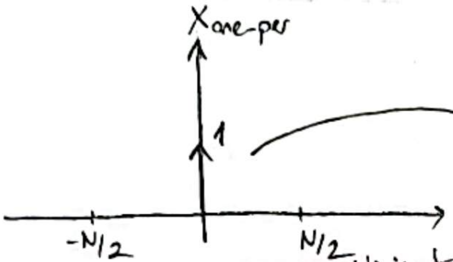
if $r \neq 1$: $$\sum_{n=0}^{N-1} \left(e^{jk\frac{2\pi}{N}}\right)^n = \frac{1-\left(e^{jk\frac{2\pi}{N}}\right)^N}{1-e^{jk\frac{2\pi}{N}}} = \frac{1-e^{jk\frac{2\pi}{N}.N}}{1-e^{jk\frac{2\pi}{N}}} = \frac{1-1}{1-e^{jk\frac{2\pi}{N}}} = 0$$

Hence,

$$\sum_{n=0}^{N-1} e^{jk\frac{2\pi}{N}n} \Rightarrow \begin{cases} N & \text{if } k = \pm mN \\ 0 & \text{else} \end{cases}$$

Example:

$x[n]$ periodic signal is given below. Find Fourier Series Representation



fourier series coefficient

$$\underbrace{x[k]}_{\text{fourier series coefficient}} = \frac{1}{N}\sum_{n,N} x[n]\, e^{-jk\frac{2\pi n}{N}} = \frac{1}{N}\sum_{n,N} \underbrace{\delta[n]}_{n=0 \Rightarrow 1} e^{-jk\frac{2\pi n}{N}} = \frac{1}{N} e^{0} = \frac{1}{N}$$

Hence, $x[n] = \sum_{k,N} x[k]\, e^{jk\frac{2\pi}{N}n}$

$$x[n] = \frac{1}{N}\sum_{k,N} e^{jk\frac{2\pi n}{N}}$$

Example:

Show that $x[k]$ is also periodic if $x[n]$ is periodic.

$x[n] = x[n+N]$

$$x[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi n}{N}}$$

$$x[k+N] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, \underbrace{e^{-j(k+N)\frac{2\pi}{N}n}}_{e^{-jk\frac{2\pi n}{N}} \cdot \underbrace{e^{-j\frac{2\pi}{\cancel{N}}.n.\cancel{N}}}_{1}}$$

extra information

Since $e^{-j(k+N)\frac{2\pi}{N}n} = e^{-jk\frac{2\pi}{N}n}$

then

$$x[k+N] = \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi n}{N}}}_{x[k]}$$

Thus $x[k+N] = x[k]$

$x[n] = x[n+N] \longleftrightarrow x[k] = x[k+N]$

Properties of Discrete time Fourier Series Coefficient

1) $\alpha x[n] + \beta y[n] \xleftrightarrow{DFSC} \alpha x[k] + \beta y[k]$

2) $x[n-n_0] \longleftrightarrow e^{\left(-jk\frac{2\pi n_0}{N}\right)} \cdot x[k]$

3) $e^{jM\left(\frac{2\pi}{N}\right) n} x[n] \longleftrightarrow x[k-M]$

4) $x^*[n] \longleftrightarrow x^*[-k]$

5) $x[-n] \longleftrightarrow x[-k]$

6) $\sum_{m,N} x[m] y[n-m] \longleftrightarrow N x[k] y[k]$

↳ periodic convolution

7) $x[n] y[n] \longleftrightarrow \underbrace{\sum_{\ell,N} x[\ell] y[k-\ell]}_{\text{periodic convolution of } x[k] \text{ and } y[k]}$

8) $\underbrace{x[n] - x[n-\ell]}_{\text{similar to } \frac{dx(t)}{dt}} \longleftrightarrow \left(1 - e^{-jk\frac{2\pi}{N}}\right) \cdot x[k]$

9) $y[n] = \begin{cases} x[n/M] & \text{if } n \text{ is multiple of } m \\ 0 & \text{else} \end{cases} \longleftrightarrow y[k] = \frac{1}{m} x[k]$

10) $\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{x[k]}{1 - e^{-jk\frac{2\pi}{N}}}$

Proof of 2

$x[n-n_0] \longleftrightarrow e^{-jk\frac{2\pi n_0}{N}} \cdot x[k]$ let $y[n] = x[n-n_0]$

$$y[k] = \frac{1}{N} \sum_{n,N} x[n-n_0] e^{-jk\frac{2\pi n}{N}}$$

let $n' = n - n_0$ and $n = n' + n_0$

$$\cancel{y[k] = \frac{1}{N}\sum_{n,N} x[n']\, e^{-jk\frac{2\pi}{N}n'} \cdot e^{-jk\frac{2\pi}{N}n_0}}$$

$$= e^{-jk\frac{2\pi n_0}{N}} \underbrace{\frac{1}{N}\sum_{n,N} x[n']\, e^{-jk\frac{2\pi n'}{N}}}_{x[k]}$$

Thus $y[k] = e^{-jk\frac{2\pi n_0}{N}} x[k]$ and $x[n-n_0] \longleftrightarrow e^{-jk\frac{2\pi}{N}n_0} x[k]$

Representation of D.T Aperiodic Signal (The DTFT)

Let $\tilde{x}[n] = \underbrace{\sum_{k,N}}_{\text{k indaksinde tek bir periyodunu N diyoruz.}} \tilde{x}[k]\, e^{jk\frac{2\pi}{N}n}$ $\quad x[n] = x[n+N] \Rightarrow \tilde{x}(n]$

$$\tilde{x}[k] = \frac{1}{N}\sum_{n,N} \tilde{x}[n] \cdot e^{-jk\frac{2\pi}{N}n}$$

$$\underbrace{N \cdot \tilde{x}[k]}_{x[\Omega]} = \sum_{n,N} \tilde{x}[n] \cdot e^{-jk\frac{2\pi}{N}n}$$

$$x[\Omega] = \sum_{k=-\infty}^{\infty} x(n]\, e^{-j\Omega n} \quad , \quad \Omega = \frac{2\pi k}{N}$$

↳ one period of $x[n]$

$w \xrightarrow{\text{büyük hali}} \Omega$

$w \to$ fourier transform for continuous

$\Omega \to$ F.T for discrete

$$x[\Omega] \xleftrightarrow{FT} x[n] \qquad \Omega_0 = \frac{2\pi}{N} \ , \ \frac{1}{N} = \frac{\Omega_0}{2\pi}$$

$$\tilde{x}[n] = \sum_{k,N} \tilde{x}[k]\, e^{jk\frac{2\pi n}{N}} = \sum_{k,N} \frac{1}{N} \times \Big[\overbrace{\frac{k 2\pi}{N}}^{\Omega}\Big] e^{jk\frac{2\pi}{N}n} ,$$

$$x[k] = \frac{x(\Omega)}{N}$$

$$= \sum_{k,N} \underbrace{\frac{\Omega_0}{2\pi} \underbrace{x\Big[\frac{k 2\pi}{N}\Big]}_{x[\Omega]}}_{x(k)} \cdot e^{jk\frac{2\pi}{N}n}$$

as $N \to \infty$ $\quad \tilde{x}[n] = x(n]$

↳ one period

$$\tilde{x}[n] = \frac{\Omega_0}{2\pi}\sum_{k,N} x[k.\Omega_0]\, e^{jk\Omega n}$$

↳ since $x[n] = x[N]$ for $N \to \infty$

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} x[\Omega]\, e^{j\Omega n}\, d\Omega}$$ Inverse F.T of D.T signal.

Pairs $\begin{cases} X[k] = \frac{1}{T} \sum x[n] e^{-jk\omega_0 n} \\ x[n] = \sum X[k] e^{jk\omega_0 n} \end{cases}$ FSC

Pairs $\begin{cases} X(\omega) = \sum x[n] e^{-j\omega n} \\ x[n] = \frac{1}{2\pi} \sum X[\omega] e^{j\omega n} \end{cases}$ FT

Thus for a periodic sequence Fourier Transform of $x[n]$

$$X[\Omega] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

inverse Fourier transform of $X[\Omega]$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} \, d\Omega$$

for the convergence of $X[\Omega]$

if $X(\Omega)$ doesn't go to ∞ then $|X(\Omega)| < \infty$

$$X(\Omega) < \infty \rightarrow \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \right| < \infty$$

$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ $x[n]$ will converge and $X[\Omega]$ exists.

Example:

$x[n] = \delta[n-n_0]$ $X[\Omega] = ?$

$$X[\Omega] = \sum_{n=-\infty}^{\infty} \delta(n-n_0) e^{-j\Omega n}$$

Thus $\left.\begin{array}{l} \delta(n-n_0) \xleftrightarrow{FT} e^{-j\Omega n_0} \\ \delta(n+n_0) \xleftrightarrow{F.T} e^{j\Omega n_0} \end{array}\right\} n > 0$

inverse F.T

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X[\Omega] e^{j\Omega n} \, d\Omega$$

$$\delta(n-n_0) = \frac{1}{2\pi} \int_{2\pi} e^{-j\Omega n_0} e^{j\Omega n} \, d\Omega$$

$$\boxed{2\pi \delta(n-n_0) = \int_{2\pi} e^{j\Omega(n-n_0)} \, d\Omega}$$

a widely used equ.

The LAPLACE TRANSFORM

Laplace transform is defined for continuous time signals and it can be thought as a generalisation of continuous time F.T.

Laplace transform of $x(t)$ is defined as,

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad s \to \text{complex variable } (s = \sigma + jw)$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + jw)t} dt$$

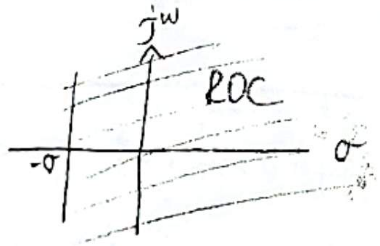
$$= \int_{-\infty}^{\infty} (x(t) e^{-\sigma t}) . e^{-jwt} dt = F.T \{ x(t) e^{-\sigma t} \}$$

Laplace transform of $x(t)$ is F.T of $x(t)e^{-\sigma t}$ for convergence of $X(s)$.

$$\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| < \infty$$

Can be satisfied for a range of σ values for which we call this region as ROC (region of convergence)

Since $(\sigma + jw)$ can be defined on s plane ROC of $X(s)$ will also be defined on s-plane



Example;

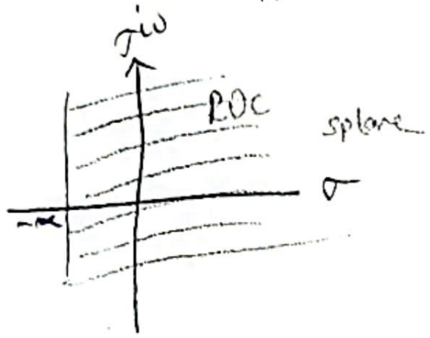
$x(t) = e^{-\alpha t} u(t) \qquad X(s) = ?$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) . e^{-st} dt$$

$$X(s) = \int_{0}^{\infty} e^{-\alpha t} e^{-st} dt = \int_{0}^{\infty} e^{-\alpha t} e^{-(\sigma + jw)t} dt$$

$$= \int_0^\infty e^{-t(\alpha+\sigma)} \cdot \underbrace{e^{-j\omega t}} dt$$

↳ $\alpha+\sigma > 0$ must be satisfied, otherwise it doesn't converge.



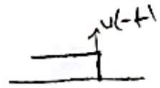
if $\alpha+\sigma>0 \Rightarrow \sigma > -\alpha$

$$Re\{s\} > -\alpha$$

$$X(s) = \int_0^\infty e^{-\alpha t} e^{-st} dt = -\frac{1}{s+\alpha} \cdot e^{-t(s+\alpha)} \Big|_0^\infty = \frac{1}{s+\alpha}$$

$\underbrace{Re\{s\}}_{\sigma} > -\alpha \qquad s = j\omega + \sigma$

Example:



$x(t) = -e^{-\alpha t} u(-t)$

$X(s) = ?$ $\quad ROC = ?$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty} -e^{-\alpha t} u(-t) e^{-st} dt = \int_{-\infty}^{0} -e^{-\alpha t} e^{-st} dt$$

$$X(s) = \int_{-\infty}^{0} -e^{-t(\alpha+s)} dt \qquad Re\{s+\alpha\} < 0$$

↳ ROC condition

$$X(s) = \frac{1}{s+\alpha} e^{-t(s+\alpha)} \Big|_{-\infty}^{0}, \qquad Re\{s\} < -\alpha$$

$X(s) = \frac{1}{s+\alpha}$ Thus,

$$\boxed{\begin{array}{ll} -e^{-\alpha t} u(-t) \xleftrightarrow{L.S} \frac{1}{s+\alpha}, & Re\{s\} < -\alpha \\ e^{-\alpha t} u(t) \xleftrightarrow{L.S} \frac{1}{s+\alpha}, & Re\{s\} > -\alpha \end{array}}$$

Example:

$x(t) = \delta(t) \qquad X(s) = ?$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty} \delta(t) e^{-st} dt = 1$$

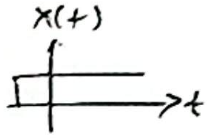
$t=0$ $\quad e^0$

Properties of ROC for Laplace Transform

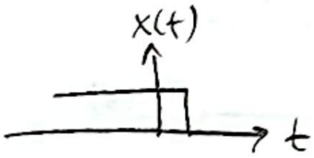
1) The ROC of $X(s)$ consists of strips parallel the $j\omega$ in the s-plane

2) For rational L.T $X(s) = \dfrac{N(s)}{D(s)}$ root of $N(s)=0$ are zeros of $X(s)$, roots of $D(s)=0$ are poles of $X(s)$. For rational L.T the ROC does not contain any poles.

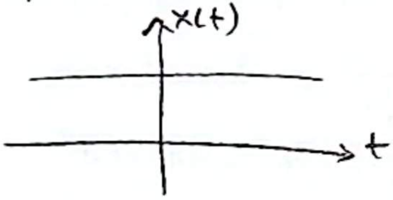
3) For a right sided ROC ; $Re\{s\} > k$ (real number)



4) For a left sided signal,
the ROC is $Re\{s\} < k$ (real number)



5) If $x(t)$ is a two sided signal then ROC will consist of a strip in the s-plane.



two sided signal
$-k_1 < Re\{s\} < k_2 \rightarrow$ The ROC

Example:

$x(t) = e^{-\alpha|t|}$. $X(s) = ?$

$$x(t) = e^{-\alpha|t|} \longrightarrow x(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ e^{-\alpha(-t)} & t < 0 \end{cases}$$

$$x(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ e^{\alpha t} & t < 0 \end{cases} \longrightarrow x(t) = e^{-\alpha t}u(t) + e^{\alpha t}u(-t)$$

$$e^{-\alpha t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \qquad \boxed{Re\{s\} > -\alpha}$$

$$e^{\alpha t}u(-t) \xrightarrow{\mathcal{L}} \int_{-\infty}^{\infty} e^{\alpha t}u(-t)e^{-st}\,dt \qquad \boxed{\alpha - s > 0} \quad \boxed{s < \alpha}$$

$$= \int_{-\infty}^{0} e^{\alpha t}e^{-st}\,dt = \int_{-\infty}^{0} e^{t(\alpha - s)}\,dt = \frac{1}{\alpha - s} = \frac{-1}{s-\alpha}$$

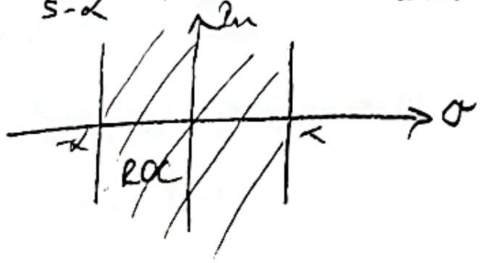
$$= \frac{1}{\alpha - s} e^{t(\alpha - s)}\Big|_{-\infty}^{0}$$

$Re\{\alpha - s\} > 0 \Rightarrow Re\{s\} < \alpha$

Thus $e^{\alpha t}u(-t) \xleftrightarrow{\mathcal{L}} \dfrac{-1}{s-\alpha}$

Hence, $e^{-\alpha|t|} \xrightarrow{\mathcal{L}} \mathcal{L}\{e^{-\alpha t}u(t)\} + \mathcal{L}\{e^{\alpha t}u(-t)\}$

$$e^{-\alpha|t|} \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} - \frac{1}{s-\alpha} \qquad ROC \Rightarrow -\alpha < R(s) < \alpha$$



Properties of Laplace Transform:

$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s) \qquad ROC = R_1$

$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s) \qquad ROC = R_2$

$x(t) \xleftrightarrow{\mathcal{L}} X(s) \qquad ROC = R$

1) Linearity: $a x_1(t) + b x_2(t) \xleftrightarrow{\mathcal{L}} a_1 X_1(s) + b X_2(s)$

$ROC = R_1 \cap R_2$

2) Time Shifting $x(t) \xleftrightarrow{\mathcal{L}} X(s) \qquad ROC = R$

$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s)$ with $ROC = R$

3) Shifting in the s-Domain

$x(t) \xleftrightarrow{\mathcal{L}} X(s) \qquad ROC = R$

$e^{st} \cdot x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0) \qquad ROC = R + Re\{s_0\}$

4) Time scaling

$x(at) \xleftrightarrow{\mathcal{L}} \dfrac{1}{|a|} X\left(\dfrac{s}{a}\right), \quad ROC = \dfrac{R}{a}$

The inverse Laplace Transform

$x(t) \xleftrightarrow{\mathcal{L}} X(s)$

$x(t) = \mathcal{L}^{-1}\{X(s)\}$

$X(s) = \mathcal{L}\{x(t)\}$

$s = \sigma + j\omega$

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt}$$

Inverse Laplace Transform

$$x(t) = \frac{1}{j2\pi} \lim_{w \to \infty} \int_{\sigma - jw}^{\sigma + jw} X(s) e^{st} \, ds$$

But since this formula is not usefull, inverse Laplace transform will be evaluated by using the L.T of some well known L.Transform

✱ 5) Convolution Property

$$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) . X_2(s)$$

$$ROC = R_1 \cap R_2$$

6) Differentiation in Time Domain

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} s.X(s) \qquad ROC \supset R$$

7) Differentiation in s-domain

$$-t\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds} \qquad ROC = R$$

8) Integration in Time Domain

$$\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} X(s) \qquad ROC \supset R \cap \{ Re\{s\} > 0 \}$$

9) The Initial and Final value Theorem

$$x(0^+) = \lim_{s \to \infty} s.X(s) \qquad \lim_{t \to \infty} x(t) = \lim_{s \to 0} s.X(s)$$

Example

$$x(t) = t.e^{-\alpha t} u(t) \qquad X(s) = ?$$

$$e^{-\alpha t} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+\alpha} \qquad Re\{s\} > -\alpha$$

$$-t.y(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dY(s)}{ds} \qquad (\text{From } P.7)$$

$$y(t) = e^{-\alpha t} u(t) \Rightarrow Y(s) = \frac{1}{s+\alpha}$$

$$\frac{dY(s)}{ds} = \frac{d(s+\alpha)^{-1}}{ds} = \frac{-1}{(s+\alpha)^2}$$

$$x(t) = t \cdot y(t) \xleftrightarrow{\mathcal{L}} -\frac{dY(s)}{ds} = X(s)$$

$$X(s) = \frac{1}{(s+\alpha)^2}, \quad Re\{s\} > -\alpha$$

Example

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt, \quad x(t) = u(t-a)$$

$$= \int_{-\infty}^{\infty} u(t-a) e^{-st} dt = \int_{a}^{\infty} e^{-st} dt$$

$$X(s) = \int_{a}^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{a}^{\infty} = \frac{e^{-sa}}{s} \quad \boxed{Re\{s\} > 0}$$

Note

$$e^{-\alpha t} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \quad Re\{s\} > -\alpha$$

$$-e^{-\alpha t} u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \quad Re\{s\} < -\alpha$$

Inverse Laplace Transform is determined regarding the ROC.

Example

$$X(s) = \frac{e^{s}}{s+1} \quad Re\{s\} > -1$$

$x(t) = ?$

* $e^{-t} u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}$ $Re\{s\} > -1$ is already known

* $y(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} Y(s)$ is known so $x(t)$ can be evaluated as;

-1 birim shift etmiş.

$$\boxed{x(t) = e^{-(t+1)} u(t+1)}$$

The Unilateral Laplace Transform

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} dt$$

↳ Lower frontier is 0 not $-\infty$.

Laplace transform of $y(t)$ is

$$Y(s) = \int_0^{\infty} y(t) e^{-st} dt = \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt$$

integration by parts $u = e^{-st}$

$dv = d\,x(t)$

$v = x(t)$

$du = -s \cdot e^{-st}$

$$= \underbrace{x(t)}_{v}\underbrace{e^{-st}}_{u}\Big|_0^{\infty} - \int x(t)(-s) \cdot e^{-st} dt$$

$$= 0 - x(0) + s\,X(s)$$

$$\boxed{\text{Thus } \frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} s\,X(s) - x(0)}$$

unilateral Laplace Transform

in a similar manner it can be shown that;

$$\boxed{L\{x''(t)\} = s^2 X(s) - s\,X(0) - x'(0)}$$

in general $L\{x^n(t)\} = s^n X(s) - s^{n-1} X(0) - s^{n-2} X(0) \ldots - x^{n-1}(0)$

Example

$x(t) = \sin t$

$\mathcal{L}\{x''(t)\} = s^2 X(s) - s\,X(0) - x'(0)$

Compute L.T of $x(t)$ (Unilateral L.T)

$x''(t) = -\sin t \longrightarrow \underbrace{x''(t) = -x(t)}$

By taking the laplace transform both sides

$\sin 0 = 0$

$s^2 X(s) - s\,X(0) - x'(0) = s^2 X(s) - 1 = -X(s)$

$\cos 0 = 1$

$$X(s) = \frac{1}{s^2+1}$$

Example

$$\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = x(t)$$

$y(0) = 3 \quad \frac{dy(0)}{dt} = -5$ } given initial conditions.

Solve the differential equ. for $x(t) = 2u(t)$

$$\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = 2u(t)$$

By L.T.

$$s^2 Y(s) - \underbrace{s\,y(0)}_{3} - \underbrace{y'(0)}_{-5} + 3sY(s) - 3\underbrace{y(0)}_{3} + 2Y(s) = \frac{2}{s}$$

$$Y(s)\left[s^2+3s+2\right] = 3s+4+\frac{2}{s}$$

$$\boxed{Y(s) = \frac{3s+4}{(s+1)(s+2)} + \frac{2}{s(s+1)(s+2)}}$$

$$Y(s) = \frac{2}{s(s+1)(s+2)} + \frac{3s+4}{(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$ Multiply both sides with $(s+1)(s+2)$

$A = 1$

$B = -1$

$C = 3$

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2} \xleftrightarrow{\mathcal{L}^{-1}} y(t) = 1 - e^{-t} + 3e^{-2t}$$

Stability of LTI Systems

$$x(t) \longrightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \longrightarrow y(t)$$

$$h(t) \xrightarrow{\mathcal{L}} H(s)$$

If it is a stable system then $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

we can decide stability using $H(s)$

$H(s)$ can be written as

$$H(s) = \frac{1}{s+a_1} + \frac{1}{s+a_2} + \dots$$ poles $-a_1, -a_2$

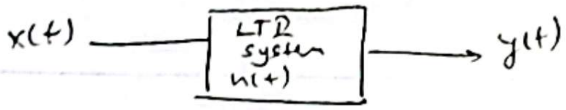
$$h(t) = e^{-a_1 t} + e^{-a_2 t} + \dots + > 0$$

if $a_n > 0$ then as $t \to \infty$ $h(t) \to \infty$ not stable system

we conclude that;

* It is stable if the poles are all on the left half of the s plane, if the poles are complex then the real parts of it must be on the left hand side for the stability.

Causality of LTI systems



If S is LTI system and if S is causal then $h(t) = 0$ for $t < 0$

$x(t) * h(t) = y(t) \longrightarrow X(s)\,H(s) = Y(s)$

$H(s) = \frac{Y(s)}{X(s)} \longrightarrow$ Transfer func. of LTI system.

$X(s) = 0 \rightarrow$ roots are the poles of $H(s)$. if ROC of $H(s)$ is to the right of the right most pole of $H(s)$ then the system S is causal else; it is non-causal.

if $H(s) = \frac{1}{s+1} + \frac{1}{s+2}$ if ROC is $Re\{s\} > -1$ then

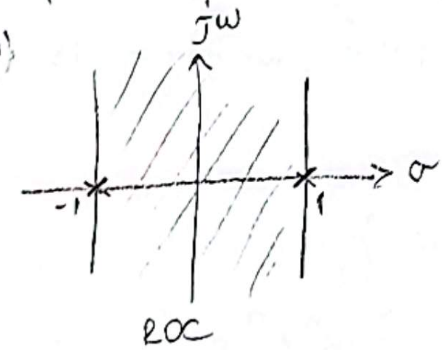
$h(t) = e^{-t}u(t) + e^{-2t}u(t)$

$h(t) = 0$ for $t < 0$ then the system is causal.

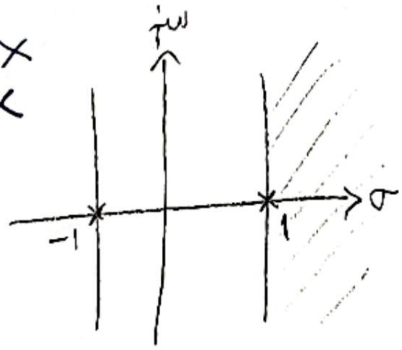
Example

Transfer function zero-pole plot of LTI systems are depicted below. Determine the stability, causality, properties of the system from the given plots.

a)



Stable ✗
Causal ✗



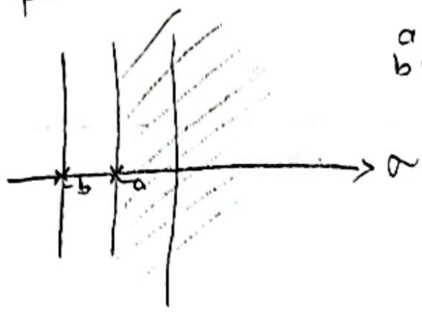
Stable ✗
Causal ✓

Example

If $H(s)$ is a stable and causal system transfer function is

$H(s) = \frac{1}{(s+a)(s+b)}$ determine arbitrary values for a and b and ROC.

poles are $-a, -b$



$a=1$
$b=1$ $\quad Re\{s\} > -1$

THE Z-TRANSFORM

Z transform is used for discrete signals.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

$X(z) \longrightarrow$ z-transform of $x[n]$

$x[n] \xleftrightarrow{z} X[z]$

$z = re^{j\Omega} \longrightarrow$ complex number

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \longrightarrow X[z] = \sum_{n=-\infty}^{\infty} x[n] (re^{j\Omega})^{-n} = X(z) = \sum_{-\infty}^{\infty} (x[n] r^{-n}) e^{-j\Omega n}$$

$$\boxed{X(z) = F.T\{x[n] r^{-n}\}}$$

Consider z transform of $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

* this summation converges for a set of z values the region in complex plane where $X(z)$ have finite summation is called ROC for $X(z)$.

Example

$x[n] = \alpha^n u[n] \quad X(z) = ? \quad ROC = ?$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (\underbrace{\alpha z^{-1}}_{r})^n = \frac{1}{1-\alpha z^{-1}} \quad |\alpha z^{-1}| < 1 \quad |z| > \alpha$$

Note

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r| < 1$$

Example

$x[n] = u[n] \quad X(z) = ?$

$\alpha^n u[n] \xleftrightarrow{z} \dfrac{1}{1-\alpha z^{-1}} \quad |z| > |\alpha|$

$u[n] \xleftrightarrow{z} \dfrac{1}{1-z^{-1}} \quad |z| > 1$

Example

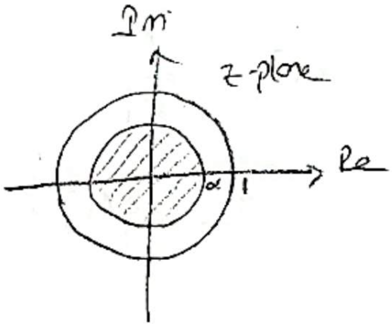
$x[n] = -\alpha^n u[-n-1], \quad X(z) = ?$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1]\, z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \alpha^{-n} z^{n} = -\sum_{n=1}^{\infty} (\alpha^{-1} z)^n = 1 - \sum_{n=0}^{\infty} (\alpha^{-1} z)^n$$

$|\alpha^{-1} z| < 1 \longrightarrow |z| < |\alpha|$

$$\boxed{X(z) = \dfrac{-\alpha^{-1} z}{1-\alpha^{-1} z}} = \dfrac{1}{1-\alpha z^{-1}}$$



$|z| < |\alpha|$
ROC

Example

$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n] \quad X(z) = ?$

$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \dfrac{1}{1-\frac{1}{2} z^{-1}} \quad |z| > \frac{1}{2}$

$ROC_1 \cap ROC_2 = |z| > \frac{1}{2}$

$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \dfrac{1}{1-\frac{1}{3} z^{-1}} \quad |z| > \frac{1}{3}$

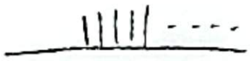
Properties of ROC of z-transform

1) The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

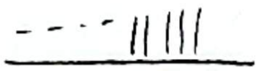
2) The ROC doesn't contain any poles.

3) If $x[n]$ is of finite domain, then the ROC is entire z-plane except possibly $z=0$ and/or $z=\infty$

4) If $x[n]$ is a right sided sequence as shown below ROC is $|z|>r_0$



If $x[n]$ is a left " " " " " " $|z|<r_0$



5) If $x[n]$ is a two sided signal, ROC is a ring in the z-plane.

Properties of the Z-Transform

$x_1[n] \xleftrightarrow{z} X_1[z]$ ROC_1

$x_2[n] \xleftrightarrow{z} X_2[z]$ ROC_2

1) Linearity

$\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{z} \alpha X_1[z] + \beta X_2[z]$ $ROC = ROC_1 \cap ROC_2$

2) Time Shifting

$x[n] \xleftrightarrow{z} X[z]$ $ROC = Rx$

$x[n-n_0] \xleftrightarrow{z} z^{-n_0} X[z]$ $ROC = Rx$

3) Frequency Shifting

$e^{j\Omega_0 n} x[n] \xleftrightarrow{z} X(e^{-j\Omega_0} z)$ $ROC = Rx$

$z_0^n x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right)$ $ROC = z_0 Rx$

4) Time Reversal

$x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right)$ $ROC = \frac{1}{Rx}$

5) Convolution Property

$$x_1[n] * x_2[n] \xleftrightarrow{z} X_1[z]\,X_2[z] \qquad ROC = ROC_1 \cap ROC_2$$

6) Differentiation in the z-plane

$$n x[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz} \qquad ROC = R_x$$

7) The Initial Value theorem

If $x[n] = 0, \quad n < 0$ then $\quad x[0] = \lim_{z \to \infty} X(z)$

Example

$x[n] = \cos(\Omega_0 n) u[n] \quad X(z) = ?$

$$\cos(\Omega_0 n) = \frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)$$

$$x[n] = \frac{1}{2}\left[e^{j\Omega_0 n} u(n) + e^{-j\Omega_0 n} u(n)\right]$$

$$\alpha^n u[n] \xleftrightarrow{z} \frac{1}{1 - \alpha z^{-1}} \qquad |z| > |\alpha|$$

$$e^{j\Omega_0 n} u[n] \xleftrightarrow{z} \frac{1}{1 - e^{j\Omega_0} z^{-1}} \qquad |z| > \underbrace{|e^{j\Omega_0}|}_{1}$$

$$e^{-j\Omega_0 n} u(n) \xleftrightarrow{z} \frac{1}{1 - e^{-j\Omega_0} z^{-1}} \qquad |z| > |e^{-j\Omega_0}|$$

Thus, $\quad X(z) = \frac{1}{2}\left[ZT\{e^{j\Omega_0 n} u(n)\} + ZT\{e^{-j\Omega_0 n} u(n)\}\right]$

$$= \frac{1}{2}\left(\frac{1}{1 - e^{j\Omega_0} z^{-1}} + \frac{1}{1 - e^{-j\Omega_0} z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1 - e^{-j\Omega_0} z^{-1} + 1 - e^{j\Omega_0} z^{-1}}{(1 - e^{j\Omega_0} z^{-1})(1 - e^{-j\Omega_0} z^{-1})}\right]$$

Example

$x[n] = \delta(n - m) \quad X(z) = ?$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n - m) z^{-n} = \sum_{n=-\infty}^{\infty} z^{-m} = \boxed{z^{-m}}$$

The Inverse Z-Transform

z transform of $x[n]$

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z = re^{j\Omega}$$

$$X[z] = FT\{x[n] r^{-n}\} \Rightarrow x[n] r^{-n} = \frac{1}{2\pi} \int_{2\pi} X(z) e^{j\Omega n} d\Omega$$

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(z) (re^{j\Omega})^n d\Omega$$

$$z = re^{j\Omega} \longrightarrow dz = jre^{j\Omega} d\Omega \qquad x[n] = \frac{1}{2\pi} \int_{2\pi} X(z) z^{n-1} \frac{1}{j} dz$$

$$d\Omega = \frac{1}{j} z^{-1} dz \qquad = \frac{1}{j2\pi} \oint X(z) z^{n-1} dz$$

$$\Rightarrow \boxed{x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz}$$

Since it is not so easy to calculate this integral everytime generally known z-transform pairs are used instead.

Example

$$X(z) = \frac{3 - \frac{5}{16} z^{-1}}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)} \qquad |z| > \frac{1}{3} \qquad x[n] = ?$$

$$X(z) = \frac{3 - \frac{5}{16} z^{-1}}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)} = \frac{A}{\left(1 - \frac{1}{4} z^{-1}\right)} + \frac{B}{\left(1 - \frac{1}{3} z^{-1}\right)}$$

let $z^{-1} = x$ $\quad A = \cdots$ $\quad B = \cdots$

$$\alpha^n u[n] \xleftrightarrow{z} \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$

$x[n] = z^{-1}[X(z)]$ which results in

$$x[n] = A\left(\frac{1}{4}\right)^n u[n] + B\left(\frac{1}{3}\right)^n u[n]$$

while finding the inverse z-transform, we should pay attention to the given ROC, for instance;

$$-\alpha^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}}, \quad |z|<|\alpha|$$

$$\alpha^n u[n] \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}}, \quad |z|>|\alpha|$$

---

is an example if ROC is defined as

$\frac{1}{4}<|z|<\frac{1}{3}$ then $x[n]$ would be as;

$$x[n] = A\left(\frac{1}{4}\right)^n u[n] - B'\left(\frac{1}{3}\right)^n u[-n-1]$$

Example

$x[n] = \alpha^{|n|}$, $\alpha<1$

$X(z) = ?$

$$x[n] = \alpha^{|n|} \qquad x[n] = \alpha^n u[n] + \alpha^{-n} u[-n-1]$$

$$\downarrow z \qquad\qquad \downarrow z$$

$$\frac{1}{1-\alpha z^{-1}} \qquad\qquad \frac{1}{1-\alpha^{-1} z^{-1}}$$

$$|z|>|\alpha| \qquad\qquad |z|<|\alpha^{-1}|$$

By intersection $|\alpha|<z<\left|\frac{1}{\alpha}\right|$ ROC

Linear Time Invariant Systems

$x[n] \longrightarrow$ [LTI systems $h[n]$] $\longrightarrow y[n]$

$\downarrow$

impulse response of an LTI system.

$$y[n] = x[n] * h[n] \longrightarrow y[z] = X[z]H[z]$$

$$H[z] = \frac{y[z]}{X[z]}$$

$\downarrow$

Transfer func. of LTI system

By checking the poles of the transfer func. we can decide whether LTI system is stable, causal or not.

$$\boxed{\begin{aligned} &\text{If } H(z) = \frac{1}{1-p_1 z^{-1}} + \frac{1}{1-p_2 z^{-1}} + \cdots \\ &\text{Then } h[n] = p_1^n u[n] + p_2^n u[n] + \cdots \end{aligned}}$$

If LTI system is stable, then

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

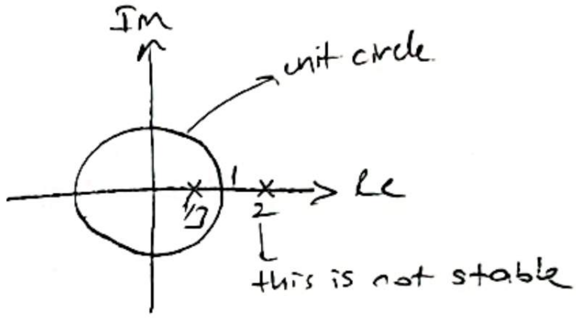
The above condition is satisfied if $|P_1| < 1$, $|P_2| < 1$ which means that all the poles should be inside the unit circle for the stability LTI system.

Example

$$H(z) = \frac{1}{(1-2z^{-1})(1-\frac{1}{3}z^{-1})}$$

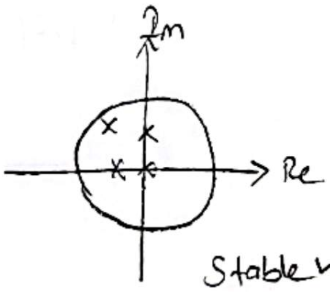
$|z| > 2$ is it a stable system or not?

Poles at $z=2$ $z=1/3$

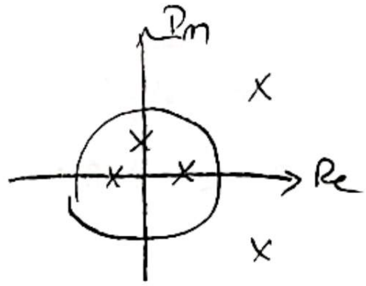


Example

For $H_1(z)$ and $H_2(z)$ decide the stability



Stable ✓



unstable X

Causality

Causality of a system is also checked using the transfer function $H(z)$ of the system such that;

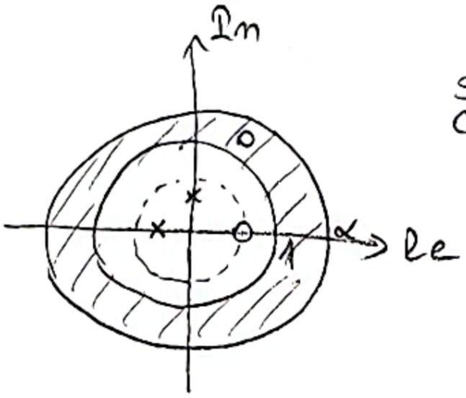
$$H[z] = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$ if a system is causal then $h[n]=0$ for $n<0$

$$H[z] = h[0]z^0 + h[1]z^{-1} + h[2]z^{-2} + \dots$$
$$= 0 + az^{-1} + bz^{-2} + cz^{-3} + \dots$$

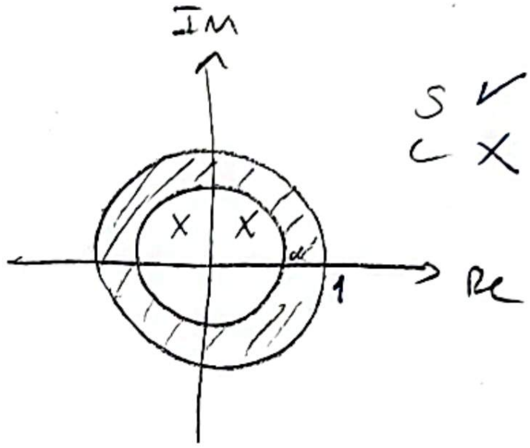
*· ROC for $H[z]$ is in the form $|z| > r_0$ thus, if ROC for only $H[z]$ is in the form $|z| > r_0$ the system is also causal.

Example

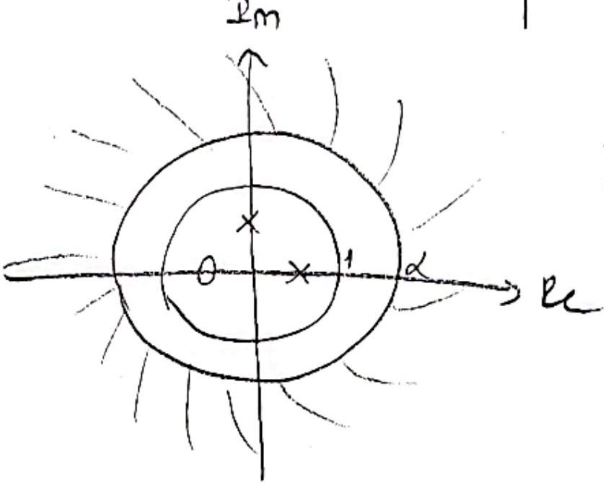
Check the given LTI systems for stability and causality.



S ✗
C ✗



S ✓
C ✗



S ✗
C ✓

Property

If the ROC of $X[z]$ contains the unit circle inside, then F.T of $x[n]$ also exists; otherwise although z-transform exists, its F.T doesnot exist!